Minimal Surface

Computational Conformal Geometry: Final project

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Outline

Section 0: Motivation

- First, I studied some discrete scheme for Gaussian curvature. However, this topic was been reported.
- Recently, Prof. Tusi gave me some paper[1] about discrete geometry for crystal. Hence, I picked up some interesting topic to report.

[1] Kotani, A discrete surface theory

Section -1: Original Study

Discrete Scheme for Gaussian Curvature. Gaussian Bonnet view point

$$K^{(1)}(\mathbf{p}) = \frac{3\left(2\pi - \sum \theta_j\right)}{A(\mathbf{p})}$$

$$K^{(2)}(\mathbf{p}) = \frac{2\pi - \sum \theta_j}{A_M(\mathbf{p})}$$

In Xu's *et al.* paper [2], a new scheme for Gaussian curvature which converges at umbilical points and regular vertexes with valence greater than 4 was introduced.

[2] Zhiqiang Xu, Guoliang Xu, Discrete schemes for Gaussian curvature and their convergence

Section 1: A Surface Theory for Graph in $\mathbb{R}^3$

1. Notation
2. The first and second functional form
3. Curvature by area-weighted average

Definition

Let $X = (V, E)$ is finite graph where V is vetices and E is edges.
Define a function $\Phi : X \to \mathbb{R}^3$. Then,

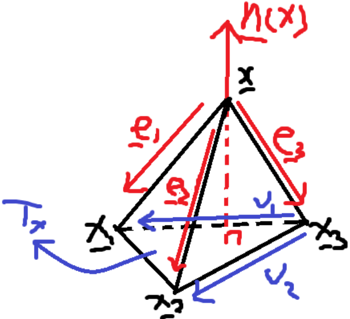
Notation

- For $e \in E$ equipped with interval $[0, 1]$. Denote $o(e)$ and $t(e)$ is origin and terminus of e.
- Set $\Phi(e) = \{t\Phi(o(e)) + (1 - t)\Phi(t(e)) : t \in [0, 1]\}$. Denote $\Phi(e)$ as $\underline{e} \in \mathbb{R}^3$

Tangent plane

Let $\Phi : X \to \mathbb{R}^3$ and for each $x \in V$ it is 3-valent. Set $E_x = \{e_1, e_2, e_3\}$.
Denote tangent plane at $\Phi(x)$ as T_x which normal vector is

$$\underline{n}(x) := \frac{(\underline{e}_1 - \underline{e}_3) \times (\underline{e}_2 - \underline{e}_3)}{|(\underline{e}_1 - \underline{e}_3) \times (\underline{e}_2 - \underline{e}_3)|}$$
$$= \frac{\underline{e}_1 \times \underline{e}_2 + \underline{e}_2 \times \underline{e}_3 + \underline{e}_3 \times \underline{e}_1}{|\underline{e}_1 \times \underline{e}_2 + \underline{e}_2 \times \underline{e}_3 + \underline{e}_3 \times \underline{e}_1|}$$



The first fundamental form

Denote $x_i = t(e_i)$ and $\underline{x}_i = \Phi(t(e_i))$.

Set $\underline{v}_i = \underline{e}_i - \underline{e}_3 = \underline{x}_i - \underline{x}_3$ Then, first fundamental form at x is

$$\mathrm{I}(x) = \begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} \langle \sigma_u, \sigma_u \rangle & \langle \sigma_u, \sigma_v \rangle \\ \langle \sigma_v, \sigma_u \rangle & \langle \sigma_v, \sigma_v \rangle \end{bmatrix}$$

$$= \begin{bmatrix} \langle \underline{v}_1, \underline{v}_1 \rangle & \langle \underline{v}_1, \underline{v}_2 \rangle \\ \langle \underline{v}_2, \underline{v}_1 \rangle & \langle \underline{v}_2, \underline{v}_2 \rangle \end{bmatrix}$$

The second fundamental form

Moreover, define covariant derivative on X

$$\nabla_i \underline{n} = (\underline{n}_i - \underline{n}_3)^T = (\underline{n}_i - \underline{n}_3) - \langle (\underline{n}_i - \underline{n}_3),\ \underline{n} \rangle \underline{n}$$

the second fundamental form

$$\mathbb{I}(x) = \begin{bmatrix} L & M_1 \\ M_2 & N \end{bmatrix} = \begin{bmatrix} -\langle \sigma_u, \nabla_u n \rangle & -\langle \sigma_u, \nabla_v n \rangle \\ -\langle \sigma_v, \nabla_u n \rangle & -\langle \sigma_v, \nabla_v n \rangle \end{bmatrix}$$
$$= \begin{bmatrix} -\langle \underline{v}_1, \nabla_1 n \rangle & -\langle \underline{v}_1, \nabla_2 n \rangle \\ -\langle \underline{v}_2, \nabla_1 n \rangle & -\langle \underline{v}_2, \nabla_2 n \rangle \end{bmatrix}$$

Note that $\mathbb{I}$ is not symmetric.

Weingarten map and curvature

The Weingarten map $\mathcal{W} : T_x \to T_x$ is $\mathrm{I}^{-1}\mathrm{II}$. Hence, $\nabla_i \underline{n}$ can be expressed as

$$\nabla_1 \underline{n} = \frac{FM_1 - GL}{EG - F^2}\underline{v}_1 + \frac{FL - EM_1}{EG - F^2}\underline{v}_2$$
$$\nabla_2 \underline{n} = \frac{FN - GM_2}{EG - F^2}\underline{v}_1 + \frac{FM_2 - EN}{EG - F^2}\underline{v}_2$$

Definition

For the discrete surface $\Phi : X \to \mathbb{R}^3$. The mean curvature $H(x)$ and Gauss curvature $K(x)$ ar $x \in V$ is defined as

$$H(x) = \frac{1}{2}\mathrm{tr}\mathcal{W} = \frac{EN + GL - F(M_1 + M_2)}{2(EG - F^2)}$$
$$K(x) = \det \mathcal{W} = \frac{LN - M_1 M_2}{EG - F^2}$$

Revisit curvature by area-weight average (1)

The $H(x)$ and $K(x)$ can be rewritten by the area-weighted average of the three curvatures around the vertex x.

Let $(\alpha, \beta) \in \{(1,2), (2,3), (3,1)\}$. Let triangle $\triangle_{\alpha\beta}$ is $\triangle(\underline{x}_0, \underline{x}_\alpha, \underline{x}_\beta)$, where $\underline{x}_0 = \mathrm{Proj}(\Phi(x))$.

$$\mathrm{I}_{\alpha\beta}(x) = \begin{bmatrix} \langle \nabla_\alpha \Phi, \nabla_\alpha \Phi \rangle & \langle \nabla_\alpha \Phi, \nabla_\beta \Phi \rangle \\ \langle \nabla_\beta \Phi, \nabla_\alpha \Phi \rangle & \langle \nabla_\beta \Phi, \nabla_\beta \Phi \rangle \end{bmatrix}$$

$$\mathrm{II}_{\alpha\beta}(x) = \begin{bmatrix} -\langle \nabla_\alpha \Phi, \nabla_\alpha n \rangle & -\langle \nabla_\alpha \Phi, \nabla_\beta \underline{n} \rangle \\ -\langle \nabla_\beta \Phi, \nabla_\alpha \underline{n} \rangle & -\langle \nabla_\beta \Phi, \nabla_\beta \underline{n} \rangle \end{bmatrix}$$

where $\nabla_e \Phi = (\Phi(e))^T = \underline{e} - \langle \underline{e}, \ \underline{n} \rangle \underline{n}$ and $\nabla_e \underline{n} = (\underline{n}(t(e)) - \underline{n}(o(e)))^T$

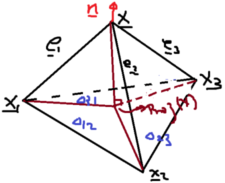
Revisit curvature by area-weight average (2)

Under this setting, the first and second fundamental form of $\triangle_{\alpha\beta}$ are

$$H(x) = \sum_{\alpha\beta} \frac{\sqrt{\det \mathrm{I}_{\alpha\beta}(x)}}{2A(x)} H_{\alpha\beta}(x)$$

$$K(x) = \sum_{\alpha\beta} \frac{\sqrt{\det \mathrm{I}_{\alpha\beta}(x)}}{2A(x)} K_{\alpha\beta}(x)$$

where $H_{\alpha\beta} = \mathrm{tr}(\mathrm{I}_{\alpha\beta}^{-1}\mathrm{II}_{\alpha\beta})$ and $K_{\alpha\beta} = \det(\mathrm{I}_{\alpha\beta}^{-1}\mathrm{II}_{\alpha\beta})$ and
$A = \frac{1}{2}|\underline{e}_1 \times \underline{e}_2 + \underline{e}_2 \times \underline{e}_3 + \underline{e}_3 \times \underline{e}_1|$



Section 2: Dirichlet Energy

1. Harmonic function
2. Laplacian operator via FEM
3. Discrete first variation formula
4. Harmonic realization

Recall: Harmonic function

Recall the Prof. Yueh's Lecture Notes, a harmonic function $f : \mathcal{M} \to \mathbb{R}$ is a minimizer of Dirichlet functional $E_D : C^\infty(\mathcal{M}) \to \mathbb{R}$ defined as

$$E_D(f) = \frac{1}{2} \int_{\mathcal{M}} \|\nabla f\|^2 dV$$

Note that, given a smooth regular surface $\sigma : \mathcal{M} \subset \mathbb{R}^2 \to \mathbb{R}^3$, its Dirichlet energy is

$$E_D(\sigma) = \frac{1}{2} \int_{\mathcal{M}} \|\nabla \sigma\|^2 dV = \int_{\mathcal{M}} \|\sigma_u\|^2 + \|\sigma_v\|^2 dV$$
$$\geq \int_{\mathcal{M}} \|\sigma_u \times \sigma_v\| dV = \mathcal{A}(\sigma)$$

- The equality holds $\Leftrightarrow$ $\|\sigma_u\| = \|\sigma_v\|$ and $\sigma_u \cdot \sigma_v = 0$
- Minimize the Dirichlet energy instead of area functional

Recall Laplacian operator: via FEM

Here, the review of Laplacian operator is follows Crane [5].

Choice of basis functions are the piecewise linear hat functions h_i, which equal one at their associated vertex and zero at all other vertices.

Given a function f such that $f(v_i) = a_i$, then we can approximate

$$f(x) = \sum_i a_i h_i(x)$$

Then we can solve the poisson equation $\Delta f = g$ by weak derivative.

$$\int_M h_i \Delta f dA = \int_M h_i g dA ,$$

we call h_i is test function.

Laplacian operator (1)

On the left hand side

$$\begin{aligned}\int_M h_i \Delta f dA &= -\int_M \nabla h_i \cdot \nabla f dA \\ &= -\int_M \nabla h_i \cdot \nabla \left(\sum_j a_j h_j \right) dA \\ &= -\sum_j a_j \int_M \nabla h_i \cdot \nabla h_j dA\end{aligned}$$

Let the matrix $L_{ij} = \int_M \nabla h_i \cdot \nabla h_j dA$. Then the equation on left hand side is

$$\begin{bmatrix} \int_M h_1 \Delta f dA \\ \vdots \\ \int_M h_{|V|} \Delta f dA \end{bmatrix} = L\mathbf{a}$$

Laplacian operator (2)

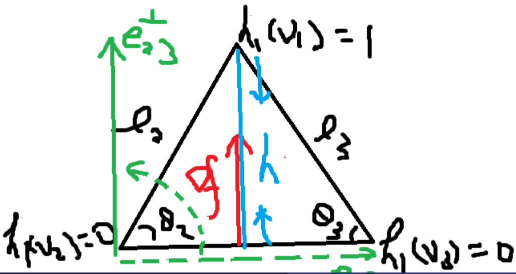
Consider function h_1 and triangle (v_1, v_2, v_3). Then, $h_1(v_1) = 1$, $h_1(v_2) = 0$ and $h_1(v_3) = 0$.

For a linear function

$$\nabla h_1 \cdot (v_1 - v_3) = 1$$

$$\nabla h_1 \cdot (v_1 - v_2) = 1$$

$$\nabla h_1 \cdot (v_2 - v_3) = 0$$



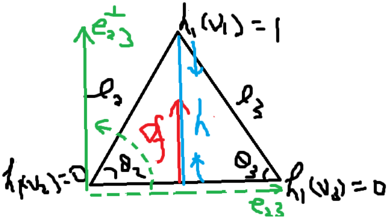
Laplacian operator (3))

Hence,

$$1 = \nabla h_1 \cdot (v_1 - v_3) = \|\nabla h_1\| l_3 \cos\left(\frac{\pi}{2} - \theta_3\right) = \|\nabla h_1\| l_3 \sin\theta_3$$

$$\|\nabla h_1\| = \frac{1}{l_3 \sin\theta_3} = \frac{1}{h}$$

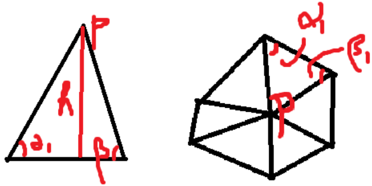
Thus, $\nabla h_1 = \frac{e_{23}^{\perp}}{2A}$, where $e_{23}^{\perp}$ is the vector from v_2 to v_3 rotated by a quarter turn in the counter-clockwise direction.



Laplacian operator (4)

There are two different cases. If $i = j$, then

$$\int_T \langle \nabla h_i, \nabla h_i \rangle dT = A \|\nabla h_i\|^2 = \frac{A}{h^2}$$
$$= \frac{h \cot \alpha + h \cot \beta}{2h}$$
$$= \frac{1}{2}(\cot \alpha + \cot \beta)$$



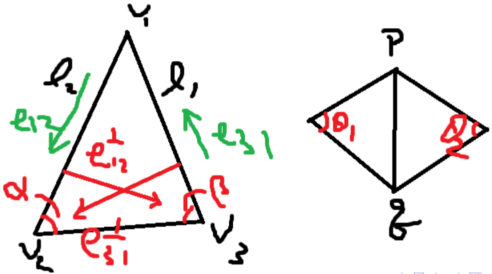
Laplacian operator (5)

If $i \neq j$, then

$$\int_T \langle \nabla h_i, \nabla h_j \rangle dT = A \langle \nabla h_i, \nabla h_j \rangle = \frac{\langle e_{31}^{\perp}, e_{12}^{\perp} \rangle}{4A} = \frac{-l_1 l_2 \cos\theta}{4A}$$

$$= \frac{-h\cos\theta}{2(h\cot\alpha + h\cot\beta)\sin\alpha\sin\beta}$$

$$= \frac{-1}{2}\cot\theta$$



Therefore,

$$\langle \nabla h_p, \nabla h_p \rangle = \frac{1}{2} \sum_i (\cot \alpha_i + \cot \beta_i)$$
$$\langle \nabla h_p, \nabla h_q \rangle = -\frac{1}{2} (\cot \theta_1 + \cot \theta_2)$$

Hence, the Laplacian matrix is

$$L_{ij} = \begin{cases} \frac{1}{2} \sum_{i \neq j} (\cot \alpha_j + \cot \beta_j), & \text{if } i = j \\ -\frac{1}{2} (\cot \alpha_j + \cot \beta_j), & \text{if } i \neq j \\ 0, & \text{otherwise} \end{cases}$$

Discrete first variation formula

In Kotani *et al.* [3], define 1-dimension Riemannian manifold with metric $ds^2 = m_E^2(e)dt^2$ on each edge $e \in E$. For the discrete surface $\Phi : X \to \mathcal{M} \subset \mathbb{R}^3$, define the energy

$$E(\Phi) = \frac{1}{2}\int_{\mathcal{M}} \|\nabla f\|^2 dV = \frac{1}{2}\sum_{e\in E} m_E(e)\int_0^1 \left\|\frac{d\Phi(e(t))}{dt}\right\|^2 dt$$

Now, the discrete first variation formula is stated as follow: Let $\Phi_e(u,t)$ be smooth variation of $\Phi(e(t)) = \Phi_e(0,t)$ and $V_e = \partial_u\Phi_e(u,t)$ and $V_e(x = t(e)) = \partial_u\Phi_e(u,0)$. Then,

$$\frac{dE(\Phi)}{dt} = -2\sum_{v\in V}\langle V(x), \sum_{e\in E_x} m_E(e)\dot{\Phi}_e(0)\rangle - \sum_{e\in E} m_E(e)\int_0^1 \langle V_e, \frac{D}{dt}\dot{\Phi}_e\rangle dt$$

Continuous version of first variation formula

Let $\Gamma(u,t)$ be variation of $\gamma(t)$. L is length of curve. Above equation is mimic to continuous version

$$\frac{dL(\Gamma)}{dt} = -\sum_i \langle V(a_i), \Delta_i \dot{\gamma} \rangle - \int_a^b \langle V, \frac{D}{dt}\dot{\gamma} \rangle dt \,,$$

where $\Delta_i \dot{\gamma} = \dot{\gamma}(a_i^+) - \dot{\gamma}(a_i^-)$.

Harmonic realization

Theorem

A piecewise smooth map is a critical map if and only if is a piecewise geodesic map, i.e. e is a geodesic for every edge e, and at each $x \in V$

$$\sum_{e \in E_x} m_E \dot{\Phi}(0) = \sum_{e \in E_x} m_E(\Phi(t(e)) - \Phi(o(e))) = 0\,,$$

where second equality is because restrict to vertex.

A piecewise geodesic map satisfying above equation is called **harmonic**.

Remark

The energy of piecewise geodesic map Φ is defined as

$$E(\Phi) = \frac{1}{2} \sum_{e \in E} m_E(e) L(e)^2$$

Section 3: Minimal Surface

1. Curvature on harmonic surface
2. Minimal surface

Revisit harmonic realization

Turn our attention back to Kotani [2].

Definition

Let $X = (V, E, m)$ be a weighted graph with weight $m : E \to (0, \infty)$. A discrete surface $\Phi : X \to \mathbb{R}^3$ is said to be harmonic with weight m if it is a harmonic realization with weight m, that is, if it satisfies

$$m(e_{x,1})\Phi(e_{x,1}) + m(e_{x,2})\Phi(e_{x,2}) + m(e_{x,3})\Phi(e_{x,3}) = 0 \tag{1}$$

for all $x \in V$, where $E_x = \{e_{x,1}, e_{x,2}, e_{x,3}\}$

Curvature on harmonic surface

Exact representation of H and K in the case of discrete harmonic surfaces is given as follows.

Theorem

Let X be 3-valent discrete harmonic surface. By the same condition, fixed $x \in V$, write $E_x = \{e_1, e_2, e_3\}$. Then, $H(x)$ and $K(x)$ is as follows

$$H(x) = \frac{m_1 + m_2 + m_3}{8A(x)^2} \sum_{(\alpha,\beta,\gamma)} \frac{\left\langle \underline{e}_\alpha, \underline{e}_\beta \right\rangle \left(\left\langle \underline{e}_\alpha, \underline{n}_\beta \right\rangle + \left\langle \underline{e}_\beta, \underline{n}_\alpha \right\rangle \right)}{m_\gamma}$$

$$K(x) = -\frac{m_1 + m_2 + m_3}{8A(x)^2} \sum_{(\alpha,\beta,\gamma)} \frac{\left\langle \underline{e}_\alpha, \underline{n}_\beta \right\rangle \left(\underline{e}_\beta, \underline{n}_\alpha \right)}{m_\gamma}$$

where $(\alpha, \beta, \gamma) \in \{(1,2,3), (2,3,1), (3,1,2)\}$ and $A = \frac{1}{2}|\underline{e}_1 \times \underline{e}_2 + \underline{e}_2 \times \underline{e}_3 + \underline{e}_3 \times \underline{e}_1|$. Note that by Equation (1), $\underline{e}_i = \nabla_i \Phi \in T_x$.

Curvature on harmonic surface: proof (1)

Proof:

- Since $m_1\underline{e}_1 + m_2\underline{e}_2 + m_3\underline{e}_3 = 0$, $\underline{e}_i = \Phi(e_i)$ lie on tangent plane T_x. Hence, $e_i = \nabla_i\Phi \in T_x$
- Since cross $\underline{e}_1$, $\underline{e}_2$ and $\underline{e}_3$ then $m_2(\underline{e}_2 \times \underline{e}_1) + m_3(\underline{e}_3 \times \underline{e}_1) = 0$ and so on,

$$\frac{1}{m_3}(\underline{e}_1 \times \underline{e}_2) = \frac{1}{m_1}(\underline{e}_2 \times \underline{e}_3) = \frac{1}{m_2}(\underline{e}_3 \times \underline{e}_1) = \kappa\underline{n}$$

where κ is a constant (parallel to $\underline{n}$).

Curvature on harmonic surface: proof (2)

Proof: Let $(\alpha, \beta) \in \{(1,2),(2,3),(3,1)\}$. The first fundamental form is

$$\mathrm{I}_{\alpha\beta}(x) = \begin{bmatrix} \langle \nabla_\alpha \Phi, \nabla_\alpha \Phi \rangle & \langle \nabla_\alpha \Phi, \nabla_\beta \Phi \rangle \\ \langle \nabla_\beta \Phi, \nabla_\alpha \Phi \rangle & \langle \nabla_\beta \Phi, \nabla_\beta \Phi \rangle \end{bmatrix} = \begin{bmatrix} \langle \underline{e}_\alpha, \underline{e}_\alpha \rangle & \langle \underline{e}_\alpha, \underline{e}_\beta \rangle \\ \langle \underline{e}_\beta, \underline{e}_\alpha \rangle & \langle \underline{e}_\beta, \underline{e}_\beta \rangle \end{bmatrix}$$

and the second fundamental form is

$$\mathrm{II}_{\alpha\beta}(x) = \begin{bmatrix} -\langle \nabla_\alpha \Phi, \nabla_\alpha n \rangle & -\langle \nabla_\alpha \Phi, \nabla_\beta \underline{n} \rangle \\ -\langle \nabla_\beta \Phi, \nabla_\alpha \underline{n} \rangle & -\langle \nabla_\beta \Phi, \nabla_\beta \underline{n} \rangle \end{bmatrix} = \begin{bmatrix} 0 & -\langle \underline{e}_\alpha, \underline{n}_\beta \rangle \\ -\langle \underline{e}_\beta, \underline{n}_\alpha \rangle & 0 \end{bmatrix}$$

where $\underline{n}_i = \underline{n}(t(e_i))$ and $\langle \underline{e}_\alpha, \underline{n}_\alpha \rangle = \langle \underline{e}_\beta, \underline{n}_\beta \rangle = 0$

Curvature on harmonic surface: proof (3)

Proof: Then, by formula of H and K

$$H_{\Delta\alpha\beta} = \frac{\left\langle \underline{e}_\alpha, \underline{e}_\beta \right\rangle \left(\left\langle \underline{e}_\alpha, \underline{n}_\beta \right\rangle + \left\langle \underline{e}_\beta, \underline{n}_\alpha \right\rangle \right)}{2\left(\left| \underline{e}_\alpha \right|^2 \left| \underline{e}_\beta \right|^2 - \left\langle \underline{e}_\alpha, \underline{e}_\beta \right\rangle^2 \right)}$$

$$K_{\Delta\alpha\beta} = -\frac{\left\langle \underline{e}_\alpha, \underline{n}_\beta \right\rangle \left\langle \underline{e}_\beta, \underline{n}_\alpha \right\rangle}{\left| \underline{e}_\alpha \right|^2 \left| \underline{e}_\beta \right|^2 - \left\langle \underline{e}_\alpha, \underline{e}_\beta \right\rangle^2}$$

Here, plug $\underline{e}_\alpha \times \underline{e}_\beta = m_\gamma \kappa \underline{n}$ and into $A^2 = \frac{1}{4}|\underline{e}_1 \times \underline{e}_2 + \underline{e}_2 \times \underline{e}_3 + \underline{e}_3 \times \underline{e}_1|^2$ then we have

$$\left| \underline{e}_\alpha \right|^2 \left| \underline{e}_\beta \right|^2 - \left\langle \underline{e}_\alpha, \underline{e}_\beta \right\rangle^2 = \|\underline{e}_\alpha \times \underline{e}_\beta\|^2 = \frac{4A(x)^2 m_\gamma}{m_1 + m_2 + m_3}$$

where $\gamma \neq \alpha, \beta$

Curvature on harmonic surface: proof (4)

Proof: Then, the results are follows

$$\frac{\sqrt{\det \mathrm{I}_{\alpha\beta}(x)}}{2A(x)} H_{\Delta\alpha\beta} = \frac{1}{4A(x)\sqrt{\det \mathrm{I}_{\alpha\beta}(x)}} \left\langle \underline{e}_\alpha, \underline{e}_\beta \right\rangle \left(\left\langle \underline{e}_\alpha, \underline{n}_\beta \right\rangle + \left\langle \underline{e}_\beta, \underline{n}_\alpha \right\rangle \right)$$

$$= \frac{m_1 + m_2 + m_3}{8A(x)^2} \cdot \frac{\left\langle \underline{e}_\alpha, \underline{e}_\beta \right\rangle \left(\left\langle \underline{e}_\alpha, \underline{n}_\beta \right\rangle + \left\langle \underline{e}_\beta, \underline{n}_\alpha \right\rangle \right)}{m_\gamma}$$

$$\frac{\sqrt{\det \mathrm{I}_{\alpha\beta}(x)}}{2A(x)} K_{\Delta\alpha\beta} = \frac{m_1 + m_2 + m_3}{8A(x)^2} \cdot \frac{\left\langle e_\alpha, \underline{n}_\beta \right\rangle \left\langle e_\beta, \underline{n}_\alpha \right\rangle}{m_\gamma}$$

Minimal surface

A discrete harmonic surface needs not be minimal *i.e.* $H = 0$, but Kotani [2] provides a sufficient condition for a harmonic surface to be minimal.

Theorem

Let $X = (V, E, m)$ be a weighted graph. A 3-valent harmonic discrete surface $\Phi : X \to \mathbb{R}^3$ is minimal if

$$\langle \underline{e}_1, \underline{e}_2 \rangle = \langle \underline{e}_2, \underline{e}_3 \rangle = \langle \underline{e}_1, \underline{e}_3 \rangle$$

Minimal surface: proof

Rewrite the mean curvature in above theorem

$$
\begin{aligned}
H(x) &= \frac{m_1+m_2+m_3}{8A(x)^2} \sum_{(\alpha,\beta,\gamma)} \frac{\langle \underline{e}_\alpha, \underline{e}_\beta \rangle \left(\langle \underline{e}_\alpha, \underline{n}_\beta \rangle + \langle \underline{e}_\beta, \underline{n}_\alpha \rangle \right)}{m_\gamma} \\
&= \frac{m_1+m_2+m_3}{8A(x)^2 m_1 m_2 m_3} \sum_{(\alpha,\beta,\gamma)} m_\alpha m_\beta \langle \underline{e}_\alpha, \underline{e}_\beta \rangle \left(\langle \underline{e}_\alpha, \underline{n}_\beta \rangle + \langle \underline{e}_\beta, \underline{n}_\alpha \rangle \right) \\
&= \frac{m_1+m_2+m_3}{8A(x)^2 m_1 m_2 m_3} \\
&\quad \sum_{(\alpha,\beta,\gamma)} \left\{ m_\alpha m_\beta \langle \underline{e}_\alpha, \underline{e}_\beta \rangle \langle \underline{e}_\beta, \underline{n}_\alpha \rangle + m_\gamma m_\alpha \langle \underline{e}_\gamma, \underline{e}_\alpha \rangle \langle \underline{e}_\gamma, \underline{n}_\alpha \rangle \right\} \\
&= \frac{m_1+m_2+m_3}{8A(x)^2 m_1 m_2 m_3} \sum_{(\alpha,\beta,\gamma)} m_\alpha \left\langle \langle e_\alpha, \underline{e}_\beta \rangle m_\beta \underline{e}_\beta + \langle \underline{e}_\gamma, \underline{e}_\alpha \rangle m_\gamma \underline{e}_\gamma, \underline{n}_\alpha \right\rangle
\end{aligned}
$$

The second equality is because reordering.

Minimal surface: proof

Since $\langle \underline{e}_1, \underline{e}_2 \rangle = \langle \underline{e}_2, \underline{e}_3 \rangle = \langle \underline{e}_1, \underline{e}_3 \rangle$, and $m_1 \underline{e}_1 + m_2 \underline{e}_2 + m_3 \underline{e}_3 = 0$ then

$$\begin{aligned} & m_\alpha \left\langle \left\langle e_\alpha, \underline{e}_\beta \right\rangle m_\beta \underline{e}_\beta + \left\langle \underline{e}_\gamma, \underline{e}_\alpha \right\rangle m_\gamma \underline{e}_\gamma, \underline{n}_\alpha \right\rangle \\ = & m_\alpha \left\langle \left\langle e_\alpha, \underline{e}_\beta \right\rangle (m_\beta \underline{e}_\beta + m_\gamma \underline{e}_\gamma), \underline{n}_\alpha \right\rangle \\ = & m_\alpha \left\langle \left\langle e_\alpha, \underline{e}_\beta \right\rangle (-m_\alpha \underline{e}_\alpha), \underline{n}_\alpha \right\rangle \\ = & 0 \end{aligned}$$

The last equality is because $\langle \underline{e}_\alpha, \underline{n}_\alpha \rangle = 0$

Summary

- Define the first and second fundamental form on 3-valent.
- Define harmonic realization
- Define the discrete curvature on harmonic surface.

Reference

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