A Quick Survey of Discrete Gaussian Curvature Algorithm

Sing-Yuan Yeh

Advisor: Prof. Meiheng Yueh

Institute of Applied Mathematical Sciences
National Taiwan University

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Gauss Bonnet Theorem

Gauss Bonnet Theorem (with boundary version)

Given surface M with piecewise smooth boundary ∂M, then

$$\int_M K dA + \int_{\partial M} \kappa_g ds + \sum_j \epsilon_j = 2\pi\chi(M)$$

where K is Gaussian curvature κ_g is geodesic curvature and ϵ_j is external angle.

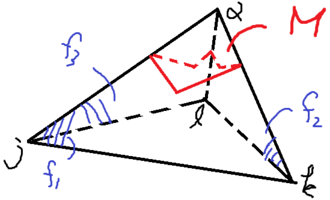
Especially, M is 2-dim surface, so $\chi(M) = 1$.

Statement of the idea

In discrete case, how do we measure the Gaussian curvature at α?

The idea is that segment a region near α and the edge is geodesic. Hence, the Gauss Bonnet theorem can be wriiten as

$$\int_M KdA + \sum_j \epsilon_j = 2\pi$$

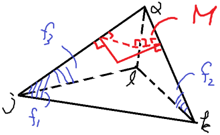


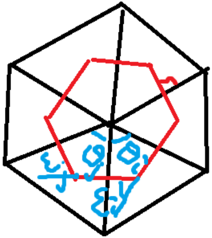
Voronoi Region

However, how to choose the area of M is important. The method is choose Voronoi region. Hence, the Gauss curvature operator is

$$\boxed{K(v_\alpha) = (2\pi - \sum_{j \in \mathcal{N}}^{\#f} \theta_j)/A}$$

where $\theta_j = \epsilon_j$.

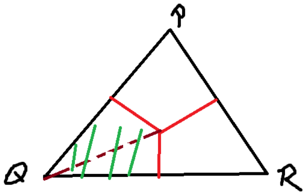




Voronoi area

The area of the green region is

$$\frac{1}{8}(|PR|^2 \cot \angle Q + |PQ|^2 \cot \angle R)$$



Voronoi area

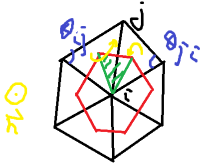
The area of the Voronoi region around the vertex i is

$$\mathcal{A}_v(v_i) = \frac{1}{8} \sum_{j \in \mathcal{N}(i)} (\cot\theta_{ij} + \cot\theta_{ji}) \|v_i - v_j\|^2$$

where $\mathcal{N}(i)$ is neighborhood of vertex i.Note the orientation of the surface.

Remark

The cotangent term is seemed like discrete Laplace Beltrami operator. Hence, we could compute it more efficiently.



Mixed area

In fact, the shape of the triangle causes the approximation inaccurate. Hence, if the triangle is non-obtuse, then we have to add some term to correct it. Please refer to Meye [1].

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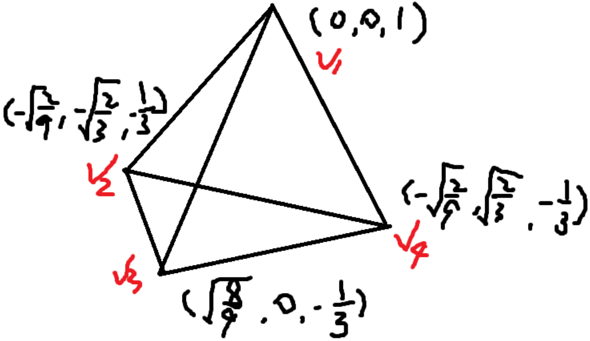
An example

In this section, an example will be introduced to explain my algorithm.

The example is tetrahedron.



Vertex and Edge matrix

$$V = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

and

$$F = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 2 \\ 2 & 4 & 3 \end{bmatrix}$$

Note the orientation.

Discrete Laplace Beltrami operator

The Discrete Laplace Beltrami operator is

$$L_{ij} = \begin{cases} \frac{-1}{2}\left(\cot\theta_{ji} + \cot\theta_{ij}\right), & \text{if } i \neq j \text{ with } ij \in \text{Edge} \\ -\sum_{k\neq i} L_{ik}, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases} .$$

Hence, first compute

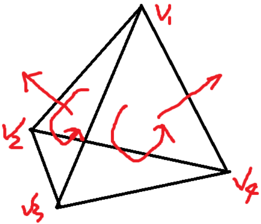
$$K = [\cot\theta_{ij}] = [\frac{e_{ki} \cdot e_{kj}}{e_{ki} \times e_{kj}}] .$$

Hence, $L = -\frac{1}{2}(K + K^T)$ except for diagonal. Therefore, $L_{ii} = -\sum_{k\neq i} L_{ik}$.

Orientation

Remark

The orientation is important. Choose $i \in F = [F_1\ \ F_2\ \ F_3]$ corresponding to $j \in [F_2\ \ F_3\ \ F_1]$ at same position. It means outward direction.



Discrete Laplace Beltrami operator

$$L = \begin{bmatrix} \sqrt{3} & \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} & \sqrt{3} & \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \sqrt{3} & \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \sqrt{3} \end{bmatrix}$$

Voronoi area

Now, according to the formula we mentioned

$$\mathcal{A}_v(v_i) = \frac{1}{8} \sum_{j \in \mathcal{N}(i)} (\cot \theta_{ij} + \cot \theta_{ji}) \| v_i - v_j \|^2 ,$$

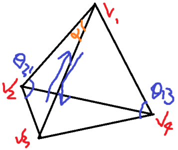
the Voronoi area is $\frac{2\sqrt{3}}{3}$, which equals to the value we compute it by intuition.

Angles

Second, compute the angles $T = [\theta_{ij}]$, where θ_{ij} can compute by cosine formula. **Note:** i, j run over vertex.
Hence, fixed vertex i, the angle around such vertex is

$$\sum_{j=1}^{\#f} \alpha_j = \sum_{j=1}^{\#f} \pi - \sum_{i} S_{ij}$$

where $S = T + T^T$. The matrix S means no matter the direction approach to v_i or leave to v_i, the angle corresponding to this edge must be sum up.



Angles

The angle $\sum_{j=1}^{\#f} \alpha_j = \pi$, which equals to the value we compute it by intuition.

Approximate Gaussian curvature by tents

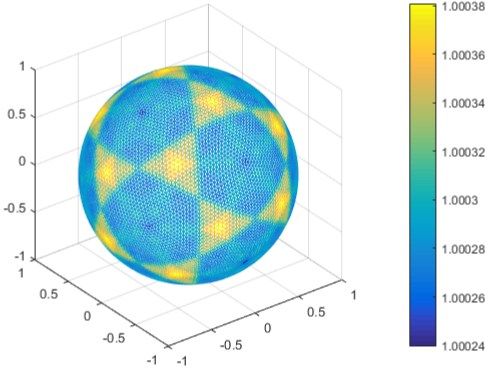
By the discrete Gaussian curvature operator, the Gaussian curvature is

$$K(v_\alpha) = (2\pi - \sum_{j \in \mathcal{N}(\alpha)}^{\#f} \theta_j)/\mathcal{A} = \frac{\pi\sqrt{3}}{2} \approx 2.72\,.$$

Outline

Result

There are 10242 vertex.



Test error

TBA

Acknowledgements

I sincerely thank Prof. Meiheng Yueh and professor's student Yunzu Hsieh for assistance.

Reference

[1] M. Meyer, M. Desbrun, P. Schroder and A. Barr, *Discrete Differential-Geometry Operators for Triangulated 2-Manifolds* (2003).