TA SESSION OF INTRODUCTION TO MATHEMATICAL ANALYSIS FOR GROUP 3 (VERSION 2)

FEBRUARY 22, 2022

(1) **Uniformly Convergence**

(a) Given a sequence of functions $\{f_n(x)\}_{n=1}^{\infty}$ which is not satisfy

$$\lim_{n\to\infty}\int_0^1 f_n(x)dx = \int_0^1 \lim_{n\to\infty} f_n(x)dx$$

(b) Find the limit of the sequence, *i.e.* $f(x) = \lim_{n\to\infty} f_n(x)$. Show that by definition.

(c) Explain why? Is $\{f_n\}$ uniformly continuous? The following theorem might be used.

Remark: $f_n \to f$ uniformly on E if and only if $\|f_n - f\| \to 0$ as $n \to \infty$. Hence, a sequence of functions uniformly converge could be viewed as a sequence converges in continuous function space.

(2) **Abel's Theorem & Differentiation and Integral**

(a) In Stewart's Calculus, we have known that

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \quad \text{on } (-1, 1)$$

Now, integrable both sides and the following is gotten,

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

Is RHS converge at $x = 1$? Is RHS equals to LHS at $x = 1$? Why?

By $\log(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$.

(b) Could we use Weierstrass M-test to state above series converge uniformly?

(c) By integration by parts, we have

$$\int_0^x \log(1+t)dt = (1+x)\log(1+x) - x$$

Now, integrable the power series or multiplication of power series, we can get that

$$(1+x)\log(1+x) - x = \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 - \cdots$$

Is RHS converge at $x = 1$? Is RHS equals to LHS at $x = 1$? Why?

(d) Back to (a), by Abel's theorem, we know power series

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

is uniformly on $[0, 1]$. However, derivative both side, and the power series is not uniformly on $[0, 1]$.