TA SESSION OF INTRODUCTION TO MATHEMATICAL ANALYSIS FOR GROUP 3 (VERSION 2)

MARCH 15, 2022

(1) Let V be the indefinite integral operator $V : L^2([0,1]) \to C^0([0,1])$, which is defined by

$$Vf(x) = \int_0^x f(t)dt\,.$$

Obviously, it is linear operator. By Holder inequality [1], exist δ such that if $|x-y| < \delta$, then

$$|Vf(x) - Vf(y)| < \int_x^y |f(t)|dt \le \|f\|_2 |y-x|$$

Hence, it is equicontinuous. Hence, we can conclude that $\overline{V(U)}$ is compact in $C^0([0,1])$. That is, V is *compact operator*.

(2) Show that $L^2([0,1])$ equipped with inner product $\langle f, g\rangle = \int_0^1 f(t)g(t)dt$ is Hilbert space.

(3) Let $\{\psi_i = \sqrt{n}\chi_{[\frac{i-1}{n},\frac{i}{n}]}\}_{i=1}^n$ and $\{\phi_i = \frac{\sqrt{n}}{n}\}_{i=1}^n$ be othornormal vector[2] Consider

$$\int_0^x f(t)\psi_i(t)dt = \begin{cases} f(\xi_i)\frac{\sqrt{n}}{n} & \text{if } \frac{i}{n} \ge x \\ 0 & \text{o.w.} \end{cases}.$$

For convenience, let

$$\Delta_i^{(n)} = \begin{cases} \frac{\sqrt{n}}{n} & \text{if } \frac{i}{n} \ge x \\ 0 & \text{o.w.} \end{cases}$$

Let

$$V_n = \sum_{i=1}^n \langle f, \psi_i\rangle \phi_i = \sum_{i=1}^n f(\xi_i)\Delta_i^{(n)}\frac{1}{\sqrt{n}}$$

which is *finite rank operator*. We could see that

$$Vf(x) = \lim_{n\to\infty} V_n f(x) = \sum_{i=1}^n f(\xi_i)\frac{1}{n}\,.$$

(4) This is *approximate property* about compact operator on Hilbert space.

[1] In your Homework.

[2] It is not basis.