REMARK OF TAYLOR THEOREM ON DECEMBER 21

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(i) If $f \in C^m$ near $x = a$, then $R_m(h) = o(h^m)$. In fact, remainder term can be written as

$$R_m(h) = \frac{h^m}{(m-1)!} \int_0^1 (1-t)^{m-1} [f^{(m)}(a+th) - f^{(m)}(a)] dt$$

(ii) If $f \in C^m$ near $x = a$ and f is $(m+1)$-times differentiable near $x = a$, then the remainder term of order m is OBVIOUSLY little o with order m and the remainder term is

$$R_m(h) = \frac{f^{(m+1)}(\xi)}{(m+1)!} h^{m+1}$$

(iii) If $f \in C^m$ near $x = a$ and moreover C^{m+1} near a, , then the remainder term of order m is OBVIOUSLY little o with order m and the remainder term is

$$R_m(h) = \frac{h^m}{(m-1)!} \int_0^1 (1-t)^{m-1} f^{(m+1)}(a+th)(a) dt$$

(iv) Continuing the statement above, $f \in C^m$ near $x = a$ and f is $(m+1)$-times differentiable at $x = a$, the remainder term of order $m+1$ is actually little o with order $m+1$, *i.e.* $R_{m+1}(h) = o(h^{m+1})$, but we could not explicitly write down $R_{m+1}(x-a)$.

(v) Back to Calculus in Stewart,

- If $f \in C^0$ near a, then
$$f(x) - f(a) = o(1) .$$
This is Taylor Theorem reduce to $m = 0$.
- If $f \in C^0$ near a, and moreover f is differentiable near a, then
$$f(x) - f(a) = f'(\xi)(x-a) = o(1)$$
- If $f \in C^0$ near a, and moreover $f \in C^1$ near a, then
$$f(x) - f(a) = \int_a^x f'(t) dt = o(1)$$
- If f only continuous at a, then
$$f(x) - f(a) = o(1)$$
e.g. Thomae's function.

Above formulas are corresponding to (i), (ii), (iii), (iv), respectively.

(vi) Let $f \in C^m$ near $x = a$ and f can be written as

$$f(x) = a_0 + a_1(x-a) + \cdots + a_m(x-a)^m + R_m(x-a) .$$

Make sure you could prove if $R_m(x-a) = o((x-a)^m)$ then $a_i = \frac{f^{(i)}(a)}{i!}$ uniquely.

(vii) Make sure you could tell the difference between T_n converge on which interval I as $n \to \infty$ and T_n converge to original f as $n \to \infty$. *Hint:* Consider the following function

$$f(x) = e^{-\frac{1}{x^2}} , \quad x = \ 0 .$$

(viii) By above example, the function and Taylor polynomial is not bijection. For instance, both $f(x)$ and $f(x) + e^{-\frac{1}{x^2}}$ have same Taylor polynomial.

(ix) Inverse the Taylor theorem is not true. That is, $f(x) = T_m(x-a) + R_m(x-a)$ does not imply $f \in C^m$ near $x = a$. *Hint:* Consider the following function

$$f(x) = \sin\left(\frac{1}{x^4}\right) e^{-\frac{1}{x^2}}\,, \quad x = \ 0\,.$$

For any m, $T_m(x) = 0$ and $R_m(x) = f(x) \in o(x^k)$, for all k. However, $f(x) \in \ C^1$ is just differentiable at $x = 0$

(x) If expand about more points? Suppose $x_0, x_1, \cdots, x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Then, for $x \in [a, b]$ exist $\xi \in (a, b)$ such that

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n)\ ,$$

where P is Lagrange interpolating polynomial as you learned in senior high school *i.e.*

$$P(x) = \sum_{k=1}^{n} f(x_k) \prod_{\substack{i=0 \\ i=\ k}}^{n} \frac{(x - x_i)}{(x_k - x_i)}\,.$$