INTRODUCTION TO RABBE'S TEST

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THEOREM 0.1. (Generalized version of Raabe's Test)
Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of positive real number.

(i) *If there exist a real number $L > 1$ and a integer $N > 0$, such that $n(1 - \frac{a_{n+1}}{a_n}) \geq L$, for all $n \geq N$, then $\sum_{n=0}^{\infty} a_n$ is convergent.*
(ii) *If there exist a integer $N > 0$, such that $n(1 - \frac{a_{n+1}}{a_n}) \leq \frac{n}{n+1}$, for all $n \geq N$, then $\sum_{n=0}^{\infty} a_n$ is divergent.*

Proof :

(i) Since $n(1 - \frac{a_{n+1}}{n}) \geq L > 1$ for $n \geq N$, then $a_n - a_{n+1} > 0$ and $n(1 - \frac{a_{n+1}}{a_n}) - 1 \geq L - 1$. Thus,

$$\begin{aligned}
&\frac{na_n - na_{n+1}}{a_n} - 1 \geq L - 1 \\
\Rightarrow &\frac{na_n - (n+1)a_{n+1} + a_{n+1}}{a_n} - 1 \geq L - 1 \\
\Rightarrow &na_n - (n+1)a_{n+1} + a_{n+1} - a_n \geq a_n(L-1) \\
\Rightarrow &na_n - (n+1)a_{n+1} \geq a_n(L-1) + (a_n - a_{n+1}) \\
\Rightarrow &\frac{na_n - (n+1)a_{n+1}}{L-1} \geq a_n + \frac{a_n - a_{n+1}}{L-1} \geq a_n.
\end{aligned}$$

Hence, for such N,

$$\begin{aligned}
\sum_{n=0}^{m} a_n &= \sum_{n=0}^{N-1} a_n + \sum_{n=N}^{m} a_n \\
&\leq \sum_{n=0}^{N-1} a_n + \sum_{n=N}^{m} \frac{na_n - (n+1)a_{n+1}}{L-1} \\
&\leq \sum_{n=0}^{N-1} a_n + \frac{1}{L-1}[\sum_{n=N}^{m} na_n - \sum_{n=N}^{m}(n+1)a_{n+1}] \\
&= \sum_{n=0}^{N-1} a_n + \frac{1}{L-1}[Na_N - (m+1)a_{m+1}] \\
&\leq \sum_{n=0}^{N-1} a_n + \frac{Na_N}{L-1}
\end{aligned}$$

Since N is fixed number, the series $\sum_{n=0}^{m} a_n$ is bounded above. Since the sequence is positive, then the sequence $\{\sum_{n=0}^{m} a_n\}_m$ is increasing. By the completeness of **R** (monotonic bounded sequence theorem), the series $\sum_{n=0}^{\infty} a_n$ is convergent.

(ii) Since $n(1-\frac{a_{n+1}}{a_n}) \leq \frac{n}{n+1}$, for $n \geq N$, then $1-\frac{a_{n+1}}{a_n} \leq \frac{1}{n+1}$. We can get $a_{n+1} \geq \frac{n}{n+1}a_n$. So, $a_{N+1} \geq \frac{N}{N+1}a_N$, $a_{N+2} \geq \frac{N+1}{N+2}a_{N+1} \geq \frac{N}{N+2}a_N$. By the same argument,

$$a_{N+i} \geq \frac{N}{N+i}a_N.$$

Hence, for such N,

$$\begin{aligned}\sum_{n=0}^{m} a_n &= \sum_{n=0}^{N-1} a_n + \sum_{n=N}^{m} a_n \\ &= \sum_{n=0}^{N-1} a_n + \sum_{i=0}^{m-i} a_{N+i} \\ &\geq \sum_{n=0}^{N-1} a_n + \sum_{i=0}^{m-N} \frac{N}{N+i}a_N \\ &\geq \sum_{n=0}^{N-1} a_n + Na_N + \sum_{n=N}^{m} \frac{1}{n}\end{aligned}$$

Since the series $\sum_n \frac{1}{n}$ is divergent, by comparison test, the series $\sum_{n=0}^{\infty} a_n$ is divergent

□

Corollary 0.2. (Courant and John version[1]of Raabe's Test) Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of positive real number.

(i) If there exist a real number $L > 1$ and a integer $N > 0$, such that $n(\frac{a_n}{a_{n+1}} - 1) \geq L$, for all $n \geq N$, then $\sum_{n=0}^{\infty} a_n$ is convergent.

(ii) If there exist a real number $L < 1$ and a integer $N > 0$, such that $n(\frac{a_n}{a_{n+1}} - 1) \leq L$, for all $n \geq N$, then $\sum_{n=0}^{\infty} a_n$ is divergent.

NOTE: Raabe's test is sometimes useful in the case $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. Hence, exchange the term of in-equation in your text book version[1] and presented in the following Corollary. In this way, if find the ratio test equal to 1, then you can try Raabe's test, without reciprocal of the ratio.

Corollary 0.3. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of positive real number.

(i) If there exist a real number $L > 1$ and a integer $N > 0$, such that $n(1-\frac{a_{n+1}}{a_n}) \geq L$, for all $n \geq N$, then $\sum_{n=0}^{\infty} a_n$ is convergent.

(ii) If there exist a real number $L < 1$ and a integer $N > 0$, such that $n(1-\frac{a_{n+1}}{a_n}) \leq L$, for all $n \geq N$, then $\sum_{n=0}^{\infty} a_n$ is divergent.

NOTE: If you use this Corollary, don't forget to find a "gap" (i.e. $|L-1| = \epsilon > 0$), such that $n(1-\frac{a_{n+1}}{a_n})$ may not approach to 1. (In fact, so do ratio test and root test. Some people misunderstand here.

[1]Courant and John: Introduction to Calculus and Analysis Vol I. p. 567