INTRODUCTION TO MATHEMATICAL ANALYSIS MIDTERM

TA: SINGYUAN YEH

1. [Courant & John] Chapter 7.2

   Let $a_n \geq 0$ for all n and fix $\epsilon > 0$. If

$$\frac{\log \frac{1}{a_n}}{\log n} > 1 + \epsilon\,,$$

show that $\sum a_n$ converge.
Hint: Compute directly,

$$\frac{\log \frac{1}{a_n}}{\log n} > 1 + \epsilon$$
$$\log \frac{1}{a_n} > \log n^{1+\epsilon}$$
$$a_n < \frac{1}{n^{1+\epsilon}}$$

By comparison test, you can prove it.

2. [Courant & John] Chapter 7.5

   Let $a_k \in \mathbb{R}$ be sequence satisfy $\limsup_{k\to\infty} |a_k|^{\frac{1}{k}} < 1$. Show that $\sum a_k$ converge absolutely.
Hint: Let $\limsup |a_k|^{\frac{1}{k}} = r < 1$, i.e.

$$\lim_{m\to\infty} \sup\{|a_k|^{\frac{1}{k}} : k \geq m\} = r\,.$$

Take $\epsilon = \frac{1-r}{2}$, i.e. $r + \epsilon < 1$. Exist M such that if $k > M$ then $|a_k|^{\frac{1}{k}} < r + \epsilon$. That is,

$$|a_k| < (r+\epsilon)^k\,.$$

Since $r+\epsilon < 1$, $\sum_{k=M}^{\infty}(r+\epsilon)^k$ converge, which implies $\sum_{k=1}^{\infty}(r+\epsilon)^k$ converge. By comparison test,

$$\sum_{k=1}^{\infty} a_k$$

converge.

3. [Courant & John] Chapter 3.15

   For what values of s is the following integral convergent?

$$\int_0^{\infty} \frac{\sin x}{x^s} dx$$

Hint: Write down integral as

$$\int_0^\infty \frac{\sin x}{x^s} dx = \int_0^1 \frac{\sin x}{x^s} dx + \int_1^\infty \frac{\sin x}{x^s} dx \,.$$

Since $\frac{\sin x}{x} \geq 0$ for $x \in [0, 1]$, by ratio test,

$$\lim_{x\to 0} \frac{\sin x/x}{1/x^{s-1}} = 1 > 0$$

Hence, both $\int_0^1 \frac{\sin x}{x^s} dx$ and $\int_0^1 \frac{1}{x^{s-1}} dx$ have same convergent behavior. Thus, they converge when $s < 2$ and divergent when $s \geq 2$.
On the other hands, Since $\lim_{x\to\infty} \frac{\sin x}{x^s}$ doesn't exist when $s < 0$, $\int_1^\infty \frac{\sin x}{x^s} dx$ diverge if $s < 0$.
Moreover,

$$\int_1^\infty \frac{\sin x}{x^s} dx = \frac{-\cos x}{x^s}\bigg|_1^\infty - \int_1^\infty \frac{s\cos x}{x^{s+1}} dx$$

Focus on

$$\left| \int_1^\infty \frac{s\cos x}{x^{s+1}} dx \right| \leq s \int_1^\infty \frac{|\cos x|}{x^{s+1}} dx \leq \int_1^\infty \frac{1}{x^{s+1}} dx$$

which converge when $s + 1 > 1$, $s > 0$. Therefore,

$$\int_0^\infty \frac{\sin x}{x^s} dx$$

converge if $0 < s < 2$.

4. Marsden & Hoffman

Show the following series converge by integral test

$$\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n \right)$$

5. [Folland] Chapter 2

Let sequence $\{x_n\}_{n=1}^\infty$ in $\mathbb{R}$. Show that the following definition is equivalent.

(a) Define $\limsup_{n\to\infty} x_n := \lim_{n\to\infty} \sup\{x_k : k \geq n\}$.

(b) This set E contains all subsequential limits. Define $\limsup_{n\to\infty} x_n := \sup E$.

Hint: For convenience, let $y_n = \sup\{x_k : k \geq n\}$ and $\alpha = \lim_{n\to\infty} \sup\{x_k : k \geq n\}$, $\beta = \sup E$. WLOG, we only consider $\alpha, \beta < \infty$ here.

First, claim $\alpha \geq \beta$. We have to construct a subsequence bounded below by y_n. Since y_n is supreme of $\{x_k : k \geq n\}$ for all n, there exist x_n such that $y_n - \epsilon < x_n < y_n$. Choose $\epsilon = \frac{1}{i}$ for all $i \in \mathbb{N}$. We can construct subsequence $\{x_{n_i}\}$ by

$$y_1 - 1 < x_{n_1} < y_1$$
$$y_2 - \frac{1}{2} < x_{n_2} < y_2$$
$$\vdots$$

where the index $n_i \neq n_j$ if $i \neq j$. By Sandwich theorem, $\{x_{n_i}\}$ converges to $\alpha = \lim_{i\to\infty} y_i$. However, x_{n_i} bounded above by y_i, so $\alpha \geq \beta$.
Second, claim $\alpha - \epsilon < \beta \leq \alpha$, for all ϵ. Take $r \in (\alpha - \epsilon, \alpha)$. Now, we hope to construct a subsequence converge to $[r, \alpha] \subset (\alpha - \epsilon, \alpha]$. Now, claim that exist infinitely many x_i greater than r. So, we can construct the subsequence $\{x_{n_i}\}$ by

$$\alpha - \epsilon < r < x_{n_1} < y_1$$
$$\alpha - \epsilon < r < x_{n_2} < y_2$$
$$\vdots$$

by the claim, where the index $n_i \neq n_j$ if $i \neq j$. Since the subsequence $\{x_{n_i}\}$ bounded by r and y_1, exist sub-subsequence of $\{x_{n_i}\}$ such that the sub-subsequence converges in $[r, y_1]$. However, y_i decreasing to α, so exist a subsequence converge in $[r, \alpha] \subset (\alpha - \epsilon, \alpha]$. Since ϵ is arbitrary chosen, we have $\alpha = \beta$, which the desired results follows. Finally, we have to prove the claim, do it by yourself[1].
Remark: You have to claim that there are infinitely many points to choose as subsequence, otherwise we cannot find $n_i \neq n_j$ for $i \neq j$.

6. [Courant & John] Chapter 1

Prove that the following principles are equivalent in the sense that any one can be derived as a consequence of any other.
(a) Every nested sequence of intervals with real end points contains a real number.
(b) Every bounded monotone sequence converges.
(c) Every bounded infinite sequence has at least one accumulation or limit point.
(d) Every Cauchy sequence converges.
(e) Every bounded set of real numbers has an infimum and a supremum.

7. [Courant & John] Chapter 1

Determine the set the following function continuous and discontinuous

$$g(x) = \begin{cases} 0, & x \quad \text{irrational} \\ \frac{1}{q}, & x = \frac{p}{q} \quad \text{rational in lowest terms} \end{cases}$$

8. [Lee] Chapter 2

Show the following space X is topological space.
(a) Let $d(\cdot,\cdot)$ is discrete distance and $\mathcal{T}$ is collection of all open set. Then, $X = (\mathbb{R}, \mathcal{T})$.
(b) $X = (\mathbb{R}, \{\mathbb{R}, \varnothing\})$.

[1]Please refer to G. Folland, *Advanced Calculus*.

9. Stewart

Determine the convergence (absolute convergent/conditional convergent/divergent) of following series.

(a) $\sum_{n=1}^{\infty}(-1)^n \frac{n!}{n^n}$
(b) $\sum_{n=1}^{\infty}(n^{\frac{1}{n}} - 1)$
(c) $\sum_{n=1}^{\infty} ne^{-n}$
(d) $\sum_{n=1}^{\infty} \sinh(\frac{1}{n^2})$
(e) $\sum_{n=9}^{\infty} \frac{1}{n \ln(n) \cdot (\ln(\ln(n)))^2}$

Hint:

(a) **Absolute Convergence.**
By alternative series test and ratio test.

(b) **Divergence.**
Since $n^{\frac{1}{n}} = e^{\frac{1}{n}\log(n)} \approx 1 + \frac{1}{n}\log(n)$, try to compare with $\frac{1}{n}$.

(c) **Absolute Convergence.**
By ratio test or root test.

(d) **Absolute Convergence.**
Since $\sinh(\frac{1}{n^2}) = (e^{\frac{1}{n^2}} - e^{\frac{-1}{n^2}})/2 \approx [(1 + \frac{1}{n^2}) - (1 - \frac{1}{n^2})]/2 = \frac{1}{n^2}$, try to compare with $\frac{1}{n^2}$.

(e) **Absolute Convergence.**
By integral test.

10. [Lee] Chapter 2

Consider a metric space. Show that A is open if and only if it is union of open balls.

11. [Rudin]

Suppose $f \geq 0, f$ is continuous on $[a, b]$. Exist $c \in [a, b]$ such that $f(c) \neq 0$. Prove that exist $d \neq c$ with $d \in [a, b]$ such that $f(d) \neq 0$.

12. A sequence $\{a_n\}_{n=1}^{\infty}$ is divergent if it is not convergent. Prove that the following things are equivalent.

(a) $\{a_n\}_{n=1}^{\infty}$ is divergent.
(b) For every $a \in \mathbb{R}$, there exists an $\epsilon > 0$ and a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ such that $|a_{n_k} - a| \geq \epsilon$

Hint: a ⇒ b) Let $a \in \mathbb{R}$. By definition of diverge, exist $\epsilon > 0$ for all $N > 0$ such that exist $n > N$ satisfy $|a_n - a| \geq \epsilon$. Now, we construct the subsequence $\{a_{n_k}\}$ by the following process. Fix $\epsilon > 0$. Let $n_0 = 1$. We always can find $n_1 > n_0 + 1$ such that $|a_{n_1} - a| \geq \epsilon$. We further find $n_k > n_{k-1} + 1$ such that $|a_{n_k} - a| \geq \epsilon$, for all $k \geq 1$. For arbitrary $a \in \mathbb{R}$, we can apply same argument.

b ⇒ a) Let $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$. We have $n_k \geq k$ by definition of subsequence. Then for all $N > 0$, we always can find $n_N \geq N$ but $|a - a_{n_N}| \geq \epsilon$.

13. Suppose any subsequence of $\{a_n\}$, say $\{a_{n_k}\}$ has further subsequence, $\{a_{n_{k_j}}\}$ that converges to unique a. Then $\{a_n\}$ converge.
Hint: Suppose $\{a_n\}$ diverge. By Question 12, exist subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that given an $\epsilon > 0$ then $|a_{n_k} - a| \geq \epsilon$, which lead a contradiction to $\{a_{n_k}\}$ has further subsequence converge to a.

14. Let $\{a_n\}_{n=1}^{\infty}$ be sequence is divergent and bounded. Show that there exists two convergent subsequences converging to two different limits.
Hint: Since $\mathbb{R}$ is complete, bounded sequence has converge subsequence $\{a_{n_k}\} \subset \{a_n\}$ with limit a. Then, since $\{a_n\}$ diverge, by Question 12, exist subsequence $\{a_{n_j}\} \subset \{a_n\}$ such that $|a_{n_j} - a| \geq \epsilon$. Again, since $\{a_{n_j}\}$ is bounded, exist converge sub-subsequence $\{a_{n_{j_\ell}}\} \subset \{a_{n_j}\}$ with limit b. Claim $a \neq b$, so $\{a_{n_k}\}$ and $\{a_{n_{j_\ell}}\}$ is what we need. Now, leave the claim to you. Do it by yourself.

15. Determine whether each of the following conditions implies the convergence of the sequence $\{x_n\}$ in a metric space X. Give a proof or an example to support your answer. Here a subsequence $\left\{x_{n_j}\right\}$ of $\{x_n\}$ is called proper if $|\mathbb{N} \backslash \{n_j, j = 1, 2, \ldots\}| = \infty$.
   (a) Every proper subsequence of $\{x_n\}$ converges.
   (b) Suppose $X \subset \mathbb{R}$ and $\{x_n\}$ is a monotonic Cauchy sequence.
   (c) $\{x_n\}$ is a Cauchy sequence and some subsequence of $\{x_n\}$ converges.
   (d) Every proper subsequence $\left\{x_{n_j}\right\}$ has a further subsubsequence $\left\{x_{n_{j(k)}}\right\}$ that converges to a common limit $p \in X$ as $k \to \infty$.
   (e) Every subsequence $\left\{x_{n_j}\right\}$ has a further subsubsequence $\left\{x_{n_{j(k)}}\right\}$ that converges as $k \to \infty$.

   Remark: Note the definition of proper subsequence. The subsequence $\{x_{n_i} : n_i = 2, 3, \cdots\}$ is not proper subsequence, because $|\mathbb{N} \setminus \{2, 3, \cdots\}|$ is finite.
   Hint:
   (a) Construct two proper subsequences which union is equal to origin sequence. We may assume two subsequence

$$\{x_{n_i} : n_i = 2i\} \quad \text{and} \quad \{x_{m_i} : m_i = 2i - 1\}$$

   Note that above two sequence are proper subsequences. Assume they converge to x and y respectively. Suppose that $x \neq y$. Let another proper subsequence $\{x_{k_i} : k_i = 3i\}$. Let $\epsilon = \frac{d(x,y)}{4}$. If i, j sufficient large, $d(x_{n_i}, x) < \epsilon$ and $d(x_{m_j}, y) < \epsilon$. However,

$$d(x_{k_i}, x_{k_{i+1}}) \geq d(x, y) - d(x_{k_i}, x) - d(x_{k_{i+1}}, y) > d(x, y) - \frac{d(x, y)}{4} - \frac{d(x, y)}{4}$$

   where for every i, one of $\{k_i, k_{i+1}\}$ belongs to the set $\{n_j = 2j : j \in \mathbb{N}\}$ and the other belongs to the set $\{m_j = 2j - 1 : j \in \mathbb{N}\}$, *i.e.* one is odd and the other is even. Now, we have $\lim_{i\to\infty} d(x_{k_i}, x_{k_{i+1}}) > 0$, which leads a contradiction to the proper subsequence

$\{x_{k_i} : k_i = 3i\}$ converge. Therefore, $\{x_{n_i} : n_i = 2i\}, \{x_{m_i} : m_i = 2i-1\}$ converge to the same point so the origin sequence converge, which is because of $\{x_i\} = \{x_{n_i}\} \cup \{x_{m_i}\}$.

(b) Let $X = (0,1)$ and $\{x_n = \frac{1}{n}\}$. Verify the sequence $\{x_n\}$ satisfy Cauchy sequence by yourself but x_n doesn't converge in X.

Remark:

- This is because of completeness of the space. Thus, we also can construct a rational sequence converge to irrational number, *e.g.* $a_n = \left(1 + \frac{1}{n}\right)^n$ converge to e.
- Besides rational number, every compact metric space is complete. One can use above theorem to construct incomplete space.