TA SESSION OF INTRODUCTION TO MATHEMATICAL ANALYSIS

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1. Definition convergence of the sequence of functions

   Consider two sequences of function $\{f_n(x) = x^n : x \in [0,1]\}$ and $\{g_n(x) = x^n : x \in [0, 0.9]\}$
   (a) Use the definition to show $\{f_n(x)\}$ and $\{g_n(x)\}$ pointwise converge.
   (b) Use the definition to determine $\{f_n(x)\}$ and $\{g_n(x)\}$ uniform converge or not.
   (c) Use the the following theorem to determine $\{f_n(x)\}$ and $\{g_n(x)\}$ uniform converge or not.

   THEOREM 1. *The sequence $\{f_k\}$ converges to f uniformly on S if and only if there is a sequence $\{C_k\}$ of positive constants such that $|f_k(x) - f(x)| \le C_k$ for all $\mathbf{x} \in S$ and* $\lim_{k\to\infty} C_k = 0$.

2. Practice
   (a) $f_k(x) = \frac{\sin kx}{\sqrt{k}}, x \in \mathbb{R}$.
   (b) $f_k(x) = \sin^k x, x \in [0, \pi]$.
   (c) $f_k(x) = k^{-1}e^{-x^2/k}, x \in \mathbb{R}$.
   (d) $f_k(x) = xe^{-nx^2}, x > 0$
   (e) $f_k(x) = \frac{x}{1+kx^2}, x \in \mathbb{R}$

   Hint: Yes/No, $x_k = \frac{\pi}{2}$/Yes/Yes, $x_k = \frac{1}{\sqrt{2k}}$/Yes, $x_k = \frac{\pm 1}{\sqrt{k}}$

3. Practice

Let $f_k(x) = g(x)x^k$, where g is continuous on $[0,1]$ and $g(1) = 0$. Show that $f_k \to 0$ uniformly on $[0,1]$.

4. Uniformly convergence of the sequence of functions

(a) Could you give two sequence of functions $\{f_n(x)\}$ which doesn't satisfy the following condition respectively

(i)

$$\lim_{n\to\infty}\int_0^1 f_n(x)dx = \int_0^1 \lim_{n\to\infty} f_n(x)dx$$

(ii)

$$\lim_{n\to\infty}\frac{d}{dx}f_n(x)\bigg|_{x=0} = \frac{d}{dx}\lim_{n\to\infty} f_n(x)\bigg|_{x=0}$$

(b) Show that the sequence of functions $\{f_n(x)\}$ converge pointwise.

(c) Does the sequence of functions $\{f_n(x)\}$ converge uniformly?

Hint:

$$f_n(x) = n\chi_{[0,\frac{1}{n}]}(x)\,, \quad f_n(x) = \frac{tan^{-1}nx}{n} \quad \text{and} \quad f_n(x) = \frac{1}{n^2x^2+1}$$

Definition 2. *If X is a metric space, $\mathcal{C}(X)$ will denote the set of all real-valued, continuous, bounded functions with domain X. We associate with each $f \in \mathcal{C}(X)$ its supremum norm*

$$\|f\| = \sup_{x \in X} |f(x)|$$

THEOREM 3. *$\mathcal{C}(X)$ is a complete metric space with $d(f,g) = \|f - g\|$.*

THEOREM 4. *A sequence $\{f_n\}$ converges to f with respect to the metric of $\mathcal{C}(X)$ if and only if $f_n \to f$ uniformly on X.*

Proof : By Theorem 1. □

Recall the compact in Euclidean space.

THEOREM 5. *If S is a subset of $\mathbb{R}^n$, the following are equivalent:*

(1) S is compact, i.e. every open covering of S has a finite subcovering.
(2) Every sequence of points in S has a convergent subsequence whose limit lies in S.
(3)

Recall the compact in metric space.

THEOREM 6. *If S is a subset of metric space E, the following are equivalent:*

(1) S is compact, i.e. every open covering of S has a finite subcovering.
(2) Every sequence of points in S has a convergent subsequence whose limit lies in S.
(3) S is closed and totally bounded in E.

Give an example

Example 7. *Let $\mathbb{R}$ be equipped with discrete metric. $[0,1]$ is not compact.*

How to generalize to continuos function space.

Definition 8. *A family $\mathcal{F}$ of complex functions f defined on a set E in a metric space X is said to be equicontinuous on E if for every $\epsilon > 0$ there exists a $\delta > 0$ such that*

$$|f(x) - f(y)| < \varepsilon$$

whenever $d(x,y) < \delta, x \in E, y \in E$, and $f \in \mathcal{F}$. Here d denotes the metric of X.

THEOREM 9. *Let K be a compact metric space, let S be a subset of $\mathcal{C}(K)$. Then, S is compact if and only if S is uniformly closed, pointwise bounded, and equicontinuous.*

Example 10. *Consider the sequence of function $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}} \subset C([0,1])$, which is defined by*

$$f_n(t) = \sin(2^n t)$$

Note that the $C([0,1])$ is equipped with supremum norm.

(1) *Show that $\mathcal{F}$ is bounded. Moreover, show that $\|f\| = 1$.*
(2) *Show that $\mathcal{F}$ is closed.*
(3) *Show that $\|f_n - f_m\| \geq 1$, for all $m \neq n$.*
(4) *$\mathcal{F}$ has no converge subsequence converge in $\mathcal{F}$.*
(5) *Explain why? Is $\mathcal{F}$ equicontinuous?*
(6) *[Extra] Is $\mathcal{F}$ totally bounded.*