TA SESSION OF INTRODUCTION TO MATHEMATICAL ANALYSIS

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1. Definition convergence of the sequence of functions

   Consider two sequences of function $\{f_n(x) = x^n : x \in [0,1]\}$ and $\{g_n(x) = x^n : x \in [0, 0.9]\}$
   (a) Use the definition to show $\{f_n(x)\}$ and $\{g_n(x)\}$ pointwise converge.
   (b) Use the definition to determine $\{f_n(x)\}$ and $\{g_n(x)\}$ uniform converge or not.
   (c) Use the the following theorem to determine $\{f_n(x)\}$ and $\{g_n(x)\}$ uniform converge or not.

   THEOREM 1. *The sequence $\{f_k\}$ converges to f uniformly on S if and only if there is a sequence $\{C_k\}$ of positive constants such that $|f_k(x) - f(x)| \leq C_k$ for all $x \in S$ and* $\lim_{k\to\infty} C_k = 0$.

2. Practice

   (a) $f_k(x) = \frac{\sin kx}{\sqrt{k}}, x \in \mathbb{R}$.
   (b) $f_k(x) = \sin^k x, x \in [0, \pi]$.
   (c) $f_k(x) = k^{-1}e^{-x^2/k}, x \in \mathbb{R}$.
   (d) $f_k(x) = xe^{-nx^2}, x > 0$
   (e) $f_k(x) = \frac{x}{1+kx^2}, x \in \mathbb{R}$

   Hint: Yes/No, $x_k = \frac{\pi}{2}$/Yes/Yes, $x_k = \frac{1}{\sqrt{2k}}$/Yes, $x_k = \frac{\pm 1}{\sqrt{k}}$

3. Practice

Let $f_k(x) = g(x)x^k$, where g is continuous on $[0,1]$ and $g(1) = 0$. Show that $f_k \to 0$ uniformly on $[0,1]$.

4. Uniformly convergence and integral, differenitation

(a) Could you give two sequence of functions $\{f_n(x)\}$ which doesn't satisfy the following condition respectively

(i)

$$\lim_{n\to\infty}\int_0^1 f_n(x)dx = \int_0^1 \lim_{n\to\infty} f_n(x)dx$$

(ii)

$$\lim_{n\to\infty}\frac{d}{dx}f_n(x)\bigg|_{x=0} = \frac{d}{dx}\lim_{n\to\infty} f_n(x)\bigg|_{x=0}$$

(b) Show that the sequence of functions $\{f_n(x)\}$ converge pointwise.

(c) Does the sequence of functions $\{f_n(x)\}$ converge uniformly?

Hint:

$$f_n(x) = n\chi_{[0,\frac{1}{n}]}(x)\,, \quad f_n(x) = \frac{tan^{-1}nx}{n} \quad \text{and} \quad f_n(x) = \frac{1}{n^2x^2+1}$$

THEOREM 2. *Suppose f_n is Riemann integral on $[a,b]$, for $n = 1,2,3,\ldots$, and suppose $f_n \to f$ uniformly on $[a,b]$. Then f is Riemann integral on $[a,b]$, and*

$$\int_a^b f dx = \lim_{n\to\infty}\int_a^b f_n dx$$

THEOREM 3. *Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a,b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a,b]$. If $\{f_n'\}$ converges uniformly on $[a,b]$, then $\{f_n\}$ converges uniformly on $[a,b]$, to a function f, and*

$$f'(x) = \lim_{n\to\infty} f_n'(x) \quad (a \le x \le b).$$

5. Uniformly convergence and continuous

Let $f_k(x) = x^k$ on $[0, 1]$ and $f_k(x) \to f(x)$. Is f continuous?

THEOREM 4. *Let $f_k : S \to \mathbb{R}$ be continuous functions, and suppose that $f_k \to f$ (uniformly). Then f is continuous.*

6. Definition convergence of the series of functions

Show that

$$\sum_{1}^{\infty} g_n(x) = \sum_{n=1}^{\infty} \frac{(\sin nx)^2}{n^2}$$

converges uniformly.

Hint: Let $f_k(x) = \sum_{n=1}^{k} \frac{(\sin nx)^2}{n^2}$

THEOREM 5. *Suppose $g_k : S \to \mathbb{R}$ are functions such that there exist constants M_k with $|g_k(x)| \leqslant M_k$ for all $x \in S$, and $\sum_{k=1}^{\infty} M_k$ converges. Then $\sum_{k=1}^{\infty} g_k$ converges uniformly (and absolutely).*

7. Uniformly convergence and integral, differenitation

Verify that $\int_0^x e^t dt = e^x - 1$, using $e^x = \sum_0^{\infty} x^n/n!$

THEOREM 6. *Suppose $g_k : [a, b] \to \mathbb{R}$ are continuous and $\sum_{k=1}^{\infty} g_k$ converges uniformly. Then we may interchange the order of integration and summation*

$$\int_a^b \sum_{k=1}^{\infty} g_k(x)dx = \sum_{k=1}^{\infty} \int_a^b g_k(x)dx.$$

THEOREM 7. *If the g_k are differentiable, the g_k' are continuous, $\sum_{k=1}^{\infty} g_k$ converges pointwise, and if $\sum_{k=1}^{\infty} g_k'$ converges uniformly, then*

$$\left(\sum_{k=1}^{\infty} g_k\right)' = \sum_{k=1}^{\infty} g_k'$$

Definition 8. *If X is a metric space, $\mathcal{C}(X)$ will denote the set of all real-valued, continuous, bounded functions with domain X. We associate with each $f \in \mathcal{C}(X)$ its supremum norm*

$$\|f\| = \sup_{x \in X} |f(x)|$$

THEOREM 9. *(Theorem 5.8 in [Marsden]) $\mathcal{C}(X)$ is a complete metric space with $d(f, g) = \|f - g\|$.*

THEOREM 10. *(Theorem 5.7 in [Marsden]) A sequence $\{f_n\}$ converges to f with respect to the metric of $\mathcal{C}(X)$ if and only if $f_n \to f$ uniformly on X.*

Proof : By Theorem 1. □

Recall the compact in Euclidean space.

THEOREM 11. *If S is a subset of $\mathbb{R}^n$, the following are equivalent:*

(1) S is compact, i.e. every open covering of S has a finite subcovering.
(2) Every sequence of points in S has a convergent subsequence whose limit lies in S.
(3) S is closed and bounded in E.

Recall the compact in metric space.

THEOREM 12. *If S is a subset of metric space E, the following are equivalent:*

(1) S is compact, i.e. every open covering of S has a finite subcovering.
(2) Every sequence of points in S has a convergent subsequence whose limit lies in S.
(3) S is closed and totally bounded in E.

Give an example

Example 13. *Let $\mathbb{R}$ be equipped with discrete metric. $[0, 1]$ is not compact.*

How to generalize to continuos function space.

Definition 14. *(Definition 5.4 in [Marsden]) A family $\mathcal{F}$ of complex functions f defined on a set E in a metric space X is said to be equicontinuous on E if for every $\epsilon > 0$ there exists a $\delta > 0$ such that*

$$|f(x) - f(y)| < \varepsilon$$

whenever $d(x, y) < \delta, x \in E, y \in E$, and $f \in \mathcal{F}$. Here d denotes the metric of X.

THEOREM 15. *(Theorem 5.9 in [Marsden]) Let K be a compact metric space, let S be a subset of $\mathcal{C}(K)$. Then, S is compact if and only if S is uniformly closed, pointwise bounded, and equicontinuous.*

Example 16. *Consider the sequence of function* $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}} \subset C([0,1])$*, which is defined by*

$$f_n(t) = \sin(2^n t)$$

Note that the $C([0,1])$ *is equipped with supremum norm.*

(1) *Show that* $\mathcal{F}$ *is bounded. Moreover, show that* $\|f\| = 1$.
(2) *Show that* $\mathcal{F}$ *is closed.*
(3) *Show that* $\|f_n - f_m\| \geq 1$*, for all* $m \neq n$.
(4) $\mathcal{F}$ *has no converge subsequence converge in* $\mathcal{F}$.
(5) *Explain why? Is* $\mathcal{F}$ *equicontinuous?*
(6) *[Extra] Is* $\mathcal{F}$ *totally bounded.*