1. **Equivalent definition in Rudin**[1]

   Let sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}$. Show that the following definition is equivalent.

   (a) Define $\limsup_{n\to\infty} x_n := \lim_{n\to\infty} \sup\{x_k : k \geq n\}$.

   (b) This set E contains all subsequential limits. Define $\limsup_{n\to\infty} := \sup E$.

   Hint: For convenience, let $y_n = \sup\{x_k : k \geq n\}$ and $\alpha = \lim_{n\to\infty} \sup\{x_k : k \geq n\}$, $\beta = \sup E$. WLOG, we only consider $\alpha, \beta < \infty$ here.
   First, claim $\alpha \geq \beta$. We have to construct a subsequence bounded below by y_n. Since y_n is supreme of $\{x_k : k \geq n\}$ for all n, there exist x_n such that $y_n - \epsilon < x_n < y_n$. Choose $\epsilon = \frac{1}{i}$ for all $i \in \mathbb{N}$. We can construct subsequence $\{x_{n_i}\}$ by

$$y_1 - 1 < x_{n_1} < y_1$$
$$y_2 - \frac{1}{2} < x_{n_2} < y_2$$
$$\vdots$$

   where the index $n_i \neq n_j$ if $i \neq j$. By Sandwich theorem, $\{x_{n_i}\}$ converges to $\alpha = \lim_{i\to\infty} y_i$. However, x_{n_i} bounded above by y_i, so $\alpha \geq \beta$.
   Second, claim $\alpha - \epsilon < \beta \leq \alpha$, for all ϵ. Take $r \in (\alpha - \epsilon, \alpha)$. Now, we hope to construct a subsequence converge to $[r, \alpha] \subset (\alpha - \epsilon, \alpha]$. Now, claim that exist infinitely many x_i greater than r. So, we can construct the subsequence $\{x_{n_i}\}$ by

$$\alpha - \epsilon < r < x_{n_1} < y_1$$
$$\alpha - \epsilon < r < x_{n_2} < y_2$$
$$\vdots$$

   by the claim, where the index $n_i \neq n_j$ if $i \neq j$. Since the subsequence $\{x_{n_i}\}$ bounded by r and y_1, exist sub-subsequence of $\{x_{n_i}\}$ such that the sub-subsequence converges in $[r, y_1]$. However, y_i decreasing to α, so exist a subsequence converge in $[r, \alpha] \subset (\alpha - \epsilon, \alpha]$. Since ϵ is arbitrary chosen, we have $\alpha = \beta$, which the desired results follows. Finally, we have to prove the claim, do it by yourself. Please refer to G. Folland, *Advanced Calculus*.
   Remark: You have to claim that there are infinitely many points to choose as subsequence, otherwise we cannot find $n_i \neq n_j$ for $i \neq j$.

[1] W. Rudin, *Principles of Mathematical Analysis*.