0919 AND 0926 CALCULUS TA SESSION (VERSION 2)

(i) Suppose both $\lim_{x\to c} \frac{f(x)}{g(x)}$ and $\lim_{x\to c} g(x) = 0$ exist. Show that $\lim_{x\to c} f(x)$ exists and equal to 0.

proof: The proposition of limit can be used directly. The following method is prove by precise definition of limit. Let $\epsilon > 0$ arbitrary chosen. Since $\lim_{x\to c} \frac{f(x)}{g(x)}$ exists, called M, there exists $\delta_1 > 0$ such that if $0 < |x - c| < \delta_1$ then $|\frac{f(x)}{g(x)} - M| < \frac{1}{2}\sqrt{\epsilon}$. On the other hands, since $\lim_{x\to c} g(x) = 0$ exists, there exists $\delta_2 > 0$ such that if $0 < |x - c| < \delta_2$ then $|g(x)| < \min\{\frac{1}{2}\sqrt{\epsilon}, \frac{1}{2|M|}\epsilon\}$.

Hence, take $\delta = \min\{\delta_1, \delta_2\}$. If $0 < |x - c| < \delta$, then

$$
\begin{aligned}
|f(x)| = |\frac{f(x)}{g(x)}\ g(x)| &= |(\frac{f(x)}{g(x)} - M)g(x) + Mg(x)| \\
&\leq |(\frac{f(x)}{g(x)} - M)g(x)| + |Mg(x)| \\
&\leq |\frac{f(x)}{g(x)} - M| \cdot |g(x)| + |M| \cdot |g(x)| \\
&\leq \frac{1}{2}\sqrt{\epsilon} \cdot \frac{1}{2}\sqrt{\epsilon} + |M| \cdot \frac{1}{2|M|}\epsilon < \epsilon\ .
\end{aligned}
$$

Therefore, since ϵ is arbitrary chosen, $\lim_{x\to c} f(x) = 0$.

(ii) (a) Evaluate $\lim_{x\to 0} \frac{x}{\sqrt{1+x}-\sqrt{1-x}}$

(b) Find a, $b \in \mathbb{R}$ such that $\lim_{x\to 0} \frac{\sqrt{ax+b}-2}{x} = 1$

(iii) (a) Suppose interval I is open with $c \in I$. Let function $f : I \setminus \{c\} \to \mathbb{R}$ satisfies $\lim_{x \to c} f(x)$ exists. Show that f is locally bounded near c.

(b) Suppose interval I is open with $c \in I$. Let function $f, g : I \setminus \{c\} \to \mathbb{R}$ satisfies f is locally bounded and $\lim_{x \to c} g(x) = 0$. Show that $\lim_{x \to c} f(x)g(x) = 0$

(iv) Show that $\lim_{x \to 0} \sin(\frac{1}{x})$ does not exist.

(v) Evaluate the following limit

$$\lim_{x \to 0} \frac{\tan ax}{\tan bx} \ .$$

(vi) Suppose the function $f : [0, 1] \to (0, 1)$ be continuos. Show that there exists $c \in (0, 1)$ such that $f(c) + c = 1$