1003 CALCULUS TA SESSION (VERSION 3)

(i) Let function defined as follows

$$f(x) = \begin{cases} x^3 \cos \frac{1}{x}\,, & \text{if } x > 0 \\ ax + b\,, & \text{if } x \leq 0\,. \end{cases}$$

(a) Find f' for $x \neq 0$.

(b) Determine constant a, $b \in \mathbb{R}$ such that f is differentiable at $x = 0$.

Solution:

(a)

$$f(x) = \begin{cases} 3x^2 \cos \frac{1}{x} + x \sin \frac{1}{x}\,, & \text{if } x > 0 \\ a\,, & \text{if } x \leq 0\,. \end{cases}$$

(b) Since f is differentiable which implies f is continuous at $x = 0$, $\lim_{x \to 0} f(x) = f(0) = b$. Thus, $b = 0$. On the other hands, since f is differentiable at $x = 0$,

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^3 \cos \frac{1}{x}}{x} = a\,.$$

Thus, $a = 0$.

(ii) (a) Show that if $f(x)$ is continuous at c, then $|f(x)|$ is also continuous at point c.

(b) Find an example which satisfied the following statement: $|f(x)|$ is continuous at c but $f(x)$ is also discontinuous at point c.

(iii) **Important concept for differentiable but not continuous derivative**

(a) Let function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $x = 0$. Find f such that f' is discontinuous.

(b) *[Next class with MVT]* Let function $f : (a, b) \to \mathbb{R}$ is continuous on (a, b) and differentiable at $x \neq c$. Show that if $\lim_{x \to c} f'(x) = L$, then f is differentiable at $x = c$ and $f'(c) = L$

(c) *[Extra]* Let function $f : (a, b) \to \mathbb{R}$ is continuous everywhere. Find f such that f' is not differentiable everywhere.

(iv) (a) Show that if $f(x)$ is differentiable at c, then

$$\lim_{h \to 0} \frac{f(c+h) - f(c-h)}{2h} \tag{$*$}$$

exists and equal to $f'(c)$.

(b) *[Next class with Taylor Theorem]* Show that if $f(x)$ is differentiable at c, then

$$\lim_{h \to 0} \frac{2f(c+h) - f(c) - f(c-h)}{3h}$$

exists and equal to $f'(c)$.

(c) Find an example which satisfied the following statement: the equation $(*)$ holds but f is not differentiable at point c.

(v) The Legendre function is defined by

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n \,.$$

Prove the following equation

$$(x^2-1)P_n''(x) + 2xP_n'(x) - n(n+1)P_n(x) = 0\,.$$

Hint: Refer to you textbook p.235 additional exercise 25 and 26. The **Leibniz's product rule** as follows:

$$(uv)^{(n)} = \binom{n}{0} u^{(n)}v + \binom{n}{1} u^{(n-1)}v^{(1)} + \cdots + \binom{n}{n} uv^{(n)}$$

(i) The Legendre function can be extended to more general form which is solution to schrodinger equation. In fact, the number n is orbital quantum number.

(ii) The Legendre function has the orthogonal property, so it is useful to solve numerical problem.

Solution: Claim $y = [(x^2-1)^n]^{(n)}$ is solution to $(x^2-1)y'' + 2xy' - n(n+1)y = 0$. Let $u = (x^2-1)^n$, and derivative both sides. We have $u' = n(x^2-1)^{x-1}2x$ so

$$(x^2-1)u' = 2nxu\,.$$

Finally, derivative both sides $(n+1)$-times. This process can be done by yourselves. Hence, the claim is proven. [*incomplete*]

(vi) **This is important concept to extend 1-dimensional differentiable sense to higher dimension by approximation** *i.e.* **little** o**.**

Let the function $f : [a,b] \to \mathbb{R}$ be continuous. Then, f is differentiable if and only if the following statement following:

there exists $m \in \mathbb{R}$ such that $f(c+h) = f(c) + mh + o(h)$ for h is small sufficient which satisfied $\lim_{h\to 0} \frac{o(h)}{h} = 0$. Moreover, m is unique equal to $f'(c)$.

Solution:

$\Leftarrow$) Since the definition of the function o,

$$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0} \frac{f(c)+mh+o(h)-f(c)}{h} = m + \lim_{h\to 0}\frac{o(h)}{h} = m\,.$$

Thus, above limit exist which implies f is differentiable and $f'(c) = m$.

$\Rightarrow$) Since $f'(c)$ exists so $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ exists, called m. Define a function $o : I \to \mathbb{R}$ by $o(h) = f(c+h) - f(c) - mh$. Now it's sufficient to prove $\lim_{h\to 0}\frac{o(h)}{h} = 0$, which do it by yourselves. [*incomplete*]

(vii) Make sure that you know how to compute the following limits

$$\lim_{x \to 0} \frac{\sin x}{x} \qquad \text{and} \qquad \lim_{x \to 0} \frac{1 - \cos x}{x} \, .$$

(viii) Use the definition of differentiation to derive

$$\frac{d}{dx} \sin x = \cos x \, .$$

(ix) Use the implicit differentiation theorem to derive

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}} \, .$$

(x) Compute the following differentiation

$$\frac{d}{dx} \tan x \, .$$