CALCULUS TA SESSION OCTOBER 24 (SOLUTION)

(1) Useful proposition for checking Riemann integrable

Let a function $f : [a, b] \to \mathbb{R}$ is bounded. Then f is Riemann integrable on $[a, b]$ if and only if for all $\epsilon > 0$ exist partition $\mathcal{P}$ of $[a, b]$ such that $|U(f, \mathcal{P}) - L(f, \mathcal{P})| < \epsilon$, where $U(f, \mathcal{P})$ and $L(f, \mathcal{P})$ are upper and lower Riemann sum respectively.

Solution: For convenience, denote

(i) Riemann sum $R(f, \mathcal{P}, \mathcal{T}) = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$, where partition $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ and sample set $\mathcal{T} = \{c_i : c_i \in [x_{i-1}, x_i]\}$.

(ii) Riemann upper sum $U(f, \mathcal{P}) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$, where $M_i = \max\{f(x) : x \in [x_{i-1}, x_i]\}$

(iii) Riemann lower sum $L(f, \mathcal{P}) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$, where $m_i = \min\{f(x) : x \in [x_{i-1}, x_i]\}$

It's sufficient to prove it.

$\Rightarrow$) Let $\epsilon > 0$ arbitrary chosen. Since f is Riemann integrable on $[a, b]$, exist constant $I \in \mathbb{R}$ and $\mathcal{P}$, $\mathcal{T}$ such that $|\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) - I| < \epsilon$. Hence, choose partition $\mathcal{P}_1$ and $\mathcal{P}_2$ such that $|U(f, \mathcal{P}_1) - I| < \frac{\epsilon}{2}$ and $|I - L(f, \mathcal{P}_2)| < \frac{\epsilon}{2}$. Take refinement partition $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$, so

$$\begin{aligned}|U(f, \mathcal{P}) - L(f, \mathcal{P})| &\le |U(f, \mathcal{P}) - I| + |I - L(f, \mathcal{P})| \\ &\le |U(f, \mathcal{P}_1) - I| + |I - L(f, \mathcal{P}_2)| < \epsilon\,.\end{aligned}$$

$\Leftarrow$) By definition of Riemann sum, the inequality can be observed $L(f, \mathcal{P}) \le R(f, \mathcal{P}, \mathcal{T}) \le U(f, \mathcal{P})$, for all $\mathcal{P}$ and $\mathcal{T}$. Since $|U(f, \mathcal{P}) - L(f, \mathcal{P})| < \epsilon$, choose $\mathcal{P}_1$ such that $|U(f, \mathcal{P}_1) - L(f, \mathcal{P}_1)| < 1$. Then, choose $\mathcal{P}_2$ with $\mathcal{P}_1 \subseteq \mathcal{P}_2$ such that $|U(f, \mathcal{P}_2) - L(f, \mathcal{P}_2)| < \frac{1}{2}$, which means $\mathcal{P}_2$ is refinement of $\mathcal{P}_1$. Continue this process, exist sequence $\{\mathcal{P}_1, \mathcal{P}_2, \cdots\}$ such that

$$\begin{aligned}|U(f, \mathcal{P}_1) - L(f, \mathcal{P}_1)| &< 1 \\ |U(f, \mathcal{P}_2) - L(f, \mathcal{P}_2)| &< \frac{1}{2} \\ |U(f, \mathcal{P}_3) - L(f, \mathcal{P}_3)| &< \frac{1}{3} \\ &\vdots\end{aligned}$$

Since $\mathcal{P}_{i+1}$ is refinement of $\mathcal{P}_i$, the following relation can be gotten

$$L(f, \mathcal{P}_1) \le L(f, \mathcal{P}_2) \le \cdots \le U(f, \mathcal{P}_2) \le U(f, \mathcal{P}_1)\,.$$

Hence, the $R(f, \mathcal{P}, \mathcal{T})$ converge to unique number called I.[1] Therefore, $|\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) - I| < \epsilon$ holds, which implies f is Riemann integrable on $[a, b]$.

[1] This is according a big theorem so I did not complete this proof in TA class.

(2) (a) Let a function $f : [a, b] \to \mathbb{R}$ be bounded and continuous except $c \in [a, b]$. Prove that f is Riemann integrable.

(b) Show that the indicator function $\mathbb{1}_{\mathbb{Q}}(x)$ on $[0, 1]$ is not Riemann integrable.

(c) [*DIY*] Is the following function Riemann integrable?

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & , \text{ if } x \in (0, 1] \\ 0 & , \text{ if } x = 0 \end{cases}$$

(d) [*DIY*] If f is discontinuous at finitely many points, is f Riemann integrable?

(e) [*extra*] If f is discontinuous at countable infinitely many points, is f Riemann integrable?

(f) [*extra*] If f is discontinuous at uncountable infinitely many points, is f Riemann integrable?

Solution:

(a) Let $\epsilon > 0$ arbitrary chosen. Since f is bounded on $[a, b]$, exist M and m such that $m \leq f \leq M$. Take $\eta = \frac{\epsilon}{4(M-m)}$, so the interval $I = [c - \eta, c + \eta]$ with length $\frac{\epsilon}{2(M-m)}$. Since f is Riemann integrable on $[a, b] \setminus I$, exist partition $\mathcal{P}'$ such that $|U(f, \mathcal{P}') - L(f, \mathcal{P}')| < \frac{\epsilon}{2}$. Take partition $\mathcal{P} = \mathcal{P}' \cup \{c - \eta, c + \eta\}$ on $[a, b]$.

$$\begin{aligned} |U(f, \mathcal{P}) - L(f, \mathcal{P})| &= \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \\ &< \frac{\epsilon}{2} + (\bar{M} - \bar{m})(c + \eta - (c - \eta)) \\ &< \frac{\epsilon}{2} + (\bar{M} - \bar{m}) \frac{\epsilon}{2(M - m)} \leq \epsilon \end{aligned}$$

where $\bar{M} = \max\{f(x) : x \in [c - \eta, c + \eta]\}$ and $\bar{m} = \min\{f(x) : x \in [c - \eta, c + \eta]\}$. Moreover, $|U(f, \mathcal{P}) - L(f, \mathcal{P})| < \epsilon$ and $L(f, \mathcal{P}) \leq R(f, \mathcal{P}, \mathcal{T}) \leq U(f, \mathcal{P})$. Thus, f is Riemann integrable.

(b) Note that the indicator function is defined by

$$\mathbb{1}_{\mathbb{Q}}(x) = \begin{cases} 1 & , \text{ if } x \in \mathbb{Q} \\ 0 & , \text{ if } x \notin \mathbb{Q} \end{cases}$$

Take $\epsilon = \frac{1}{2}$. For all $\mathcal{P} = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ on $[0, 1]$, there exists a rational number and irrational number in $[x_i, x_{i+1}]$. Hence, $M_i = 1$ and $m_i = 0$ for all i. Hence,

$$\begin{aligned} |U(f, \mathcal{P}) - L(f, \mathcal{P})| &= \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} 1(x_i - x_{i-1}) = \sum_{i=1}^{n} (x_i - x_{i-1}) = 1 \geq \epsilon \end{aligned}$$

Thus, f is not Riemann integrable.

(c) It's similar to 2(a). Take $\eta = \frac{\epsilon}{4}$. Cover the origin by interval $I = [0, \eta]$. Hence, f is continuous on $[\eta, 1]$, which implies f is Riemann integrable on $[\eta, 1]$. By the similar notation,

$$\begin{aligned}|U(f,\mathcal{P})-L(f,\mathcal{P})| &= \sum_{i=1}^{n}(M_i-m_i)(x_i-x_{i-1})\\ &< \frac{\epsilon}{2}+(\bar{M}-\bar{m})(\eta-0) < \frac{\epsilon}{2}+(1-(-1))\frac{\epsilon}{4} \le \epsilon\,,\end{aligned}$$

where $\bar{M}=\max\{f(x) : x\in[0,\eta]\}=1$ and $\bar{m}=\min\{f(x) : x\in[0,\eta]\}=-1$.

(3) Let a function $f : [a, b] \to \mathbb{R}$ be bounded and continuous which satified that $f(x) \geq 0$ for all $x \in [a, b]$ and exist $c \in [a, b]$ such that $f(c) > 0$. Show that

$$\int_a^b f(x)\, dx > 0\,.$$

(4) (a) Find an example that a function f has an anti-derivative on $\mathbb{R}$ but not Riemann integrable on $[-1, 1]$. (*See 2(c)*)

(b) Find an example that f is Riemann integrable on $[-1, 1]$ but f does not has an anti-derivative on $[-1, 1]$.

(5) Substitution Method

Remark: $d\left(\frac{1}{x}\right) = \frac{-1}{x^2}dx \qquad d\sqrt{x} = \frac{1}{2\sqrt{x}}dx$

(a) $\int \frac{\sqrt{1-\sqrt{x}}}{\sqrt{x}}$ dx (b) $\int \frac{2^{\frac{1}{x}}}{x^2}\, dx$

(6) Substitution Method

Remark: $d\sin x = \cos x\, dx \qquad d\tan^{-1} x = \frac{1}{1+x^2}\, dx \qquad d\log x = \frac{1}{x}dx$

(a) $\int \cot x\, dx$ (b) $\int \frac{\tan^{-1}\sqrt{x}}{\sqrt{x}(1+x)}\, dx$ (c) $\int \frac{\log\sqrt{x}}{x}\, dx$

(7) Integration by Part

Evaluate the following integration $\int \sin^{-1} x\, dx$.

(8) Integration by Part

Check you know how to compute the following integration

(a) $\int e^x \sin x \, dx$ (b) Find the recursive formula of $\int \sin^n x \, dx$.

(c) Do not use recursive formula to find the following integrations $\int \sin^2 x \, dx$, $\int \sin^3 x \, dx$, and $\int \sin^4 x \, dx$

(9) Fundamental Theorem of Calculus and Leibniz's rule

Suppose a continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfy

$$x \sin(\pi x) = \int_0^{x^2} f(t) \, dt \, .$$

Find the value $f(4)$.

(10) Given $0 < \alpha < 1$, show that $x^\alpha \leq \alpha x + (1-\alpha)$, for all $x > 0$. Hint: Let $f(x) = \alpha x + (1-\alpha) - x^\alpha$.

(11) L'Hopital rule

Define $f,\ g : [0,\infty) \to [0,\infty)$ by $f(x) = \int_0^x \cos^2(t)dt$ and $g(x) = f(x)e^{\sin x}$ for $x \geq 0$. Does the equality

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

holds? **Why?**