CALCULUS TA SESSION OCTOBER 31 (VERSION 2)

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"1051 A2 Midterm Problem 3" means that 105 first semester 微甲 group 2 midterm examination problem 3. The solution can be founded on website.

(1) Differentiable and Continuous *1051 A2 Midterm Problem 3*

Suppose that a function

$$f(x) = \begin{cases} \sin x + b\log(x+1) + c\,, & \text{if } x \geq 0 \\ e^{x^2}\,, & \text{if } x < 0 \end{cases}.$$

(a) Find b, c such that f is continuous.
(b) Find b, c such that f is differentiable.
(c) For b, c in (b), is f is continuously differentiable?

(2) Differentiable and Continuous *1041 A1 Midterm Problem 1*

Let a function

$$f(x) = \begin{cases} x^\alpha \sin\left(\frac{1}{x^\beta}\right), & x > 0 \\ 0, & x = 0 \\ \frac{\sin\left(x^\beta\right)}{1-\cos x}, & x < 0 \end{cases}$$

(a) For what values of α and β will $f(x)$ be continuous at $x = 0$?
(b) For what values of α and β will $f(x)$ be differentiable at $x = 0$?

Remark: DIY, please.

(3) Linerization and MVT *1071 A1 Midterm Problem 4*

(a) Find the linearization of $f(x) = sin^{-1}x$ at $x = 0.5$. Denote the linearization by $L(x)$.
(b) Use linear approximation to estimate $sin^{-1}(0.49)$.
(c) Let $g(x) = sin^{-1}x - L(x)$. Use the Mean Value Theorem twice to estimate $|g(0.49) - g(0.5)|$ and get an upper bound for the quantity.

Remark: The important concept which I mentioned in October 3 of approximation is shown in (c). Refer to 1071 A1 Midterm another Problem 4.

(4) Leibniz Product Formula *1041 A1 Midterm Problem 5*

Find the nth derivative of the function

$$f(x) = \frac{x^n}{1-x}.$$

Remark: I mentioned this formula in October 3. You can do it by yourself.

(5) Differentiable of inverse function *1041 A1 Midterm Problem 3*

Let $f(x)$ be a twice differentiable one-to-one function. Suppose that $f(2) = 1$, $f'(2) = 3$, $f''(2) = e$. Find the following value

$$\frac{d}{dx} f^{-1}(1) \quad \text{and} \quad \frac{d^2}{dx^2} f^{-1}(1)$$

(6) Mean Value Theorem *1031 A1 Midterm Problem 6*

Suppose that the function $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), and $0 < a < b$. If $f(a) = ka$, $f(b) = kb$ for some k, show that there exists $c \in (a, b)$ such that the tangent line of $y = f(x)$ at c passes through the origin.

Remark: DIY, please.

(7) Mean Value Theorem *1011 A1 Midterm Problem 3*

Let $a < b$. A function f in said to be a contraction on $[a, b]$ if there exists K, $0 < K < 1$, such that for all $x_1, x_2 \in [a, b]$ we have $|f(x_1) - f(x_2)| \leq K\,|x_1 - x_2|$.

(a) Show by the $\epsilon - \delta$ definition that if f is a contraction on $[a, b]$, then f is continuous on $[a, b]$.

(b) Suppose that f is continuous on $[a, b]$ and differentiable on (a, b) with $|f'(x)| \leq q$, $0 < q < 1$, for all $x \in (a, b)$. Show that f is a contraction on $[a, b]$ and has at most one fixed point on $[a, b]$. (A point $c \in [a, b]$ is called a fixed point of f if $f(c) = c$.)

(8) Mean Value Theorem *1051 A1 Midterm Problem 6*

Suppose that f is a differentiable function. If $f'(a) > 0$ and $f'(b) < 0$, explain that there exists $c \in (a, b)$ such that $f'(c) = 0$.

(9) Mean Value Theorem *1061 B Midterm Problem 2*

Prove that $\tan^{-1} y - \tan^{-1} x \leq y - x$ if $y \geq x \geq 0$.

(10) *1061 A1 Final Problem 1*

Find the following limits.

(a) Definition of Riemann Integral

$$\lim_{n\to+\infty}\left(\frac{n}{n^2+4\cdot 1^2}+\frac{n}{n^2+4\cdot 2^2}+\frac{n}{n^2+4\cdot 3^2}+\ldots+\frac{n}{5n^2}\right)=\lim_{n\to+\infty}\sum_{i=1}^{n}\frac{n}{n^2+4i^2}$$

(b) Fundamental Theorem of Calculus and Leibniz Rule

$$\lim_{h\to 0}\frac{1}{h}\int_{1-h}^{\sqrt[3]{1+h}}\sqrt{1+t^3}dt$$

(11) Fundamental Theorem of Calculus *1061 B Final Problem 1*

Let a function $f:\mathbb{R}\to\mathbb{R}$ satisfy by

$$\int_0^x f(t)dt=\int_x^1 t^2 f(t)dt+\frac{x^{16}}{8}+\frac{x^{18}}{9}+C$$

for all $x\in\mathbb{R}$, where C is constant. Find $f(x)$ and constant C.

Remark: DIY, please.

(12) Mean Value Theorem for Integral *1031 A1 Final Problem 1*

Evaluate

$$\lim_{x\to 0}\frac{\int_x^{\tan x}\sqrt{1+t^3}dt}{x^3}$$

(13) Techniques of Integration and Improper Integral *1061 A1 Final Problem 4*

Find the value of the constant c for which the integral

$$\int_0^{\infty} \frac{x^2+8}{x^3+8} - \frac{c}{\sqrt{x^2+1}} dx$$

converge. Evaluate the integral for this value of c.

(14) Improper Integral *1041 A1 Final Problem 3*

Suppose that $f(x)$ is a polynomial whose coefficients are integers, and

$$\int_0^{\infty} \frac{f(x)}{(x+1)^2\,(4x^2+1)} dx = 2\ln 2 + 1\,.$$

Find $f(x)$.

Remark: DIY, please.

(15) Sketch a Graph of a Function *1051 B Midterm Problem 7*

Let a function $y = f(x) = \frac{(x+1)^3}{x^2+2x}$

(a) Find the intervals where f is decreasing or increasing.

(b) Find the intervals where f is concave up or concave down.

(c) Find the local maximum and minimum values.

(d) Find the inflection points.

(e) Sketch the graph of $y = f(x)$.