CALCULUS TA SESSION NOVEMBER 21 (VERSION 2)

(1) $\boxed{\text{Integral Test}}$

Let a series as follows

$$\sum_{n=1}^{\infty}\left[\sin^2(n\pi)+\frac{1}{n^2}\right].$$

(i) Is $\int_1^\infty \sin^2(x\pi)+\frac{1}{x^2}\,dx$ converge?

(ii) Is $\sin^2(n\pi)+\frac{1}{n^2}$ positive for all n?

(iii) Is above series converge? Why?

(2) $\boxed{\text{Integral Test}}$

Determine whether the following series converge or not.

(i) $\sum_{n=1}^{\infty}\frac{1}{n\log n}$

(ii) $\sum_{n=1}^{\infty}\frac{1}{1+e^n}$

(iii) $[DIY]$ $\sum_{n=1}^{\infty}\frac{1}{n^p}$

(3) $\boxed{\text{Ratio Test and Root Test}}$

Determine whether the following series converge or not.

(i) $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$

(ii) $\sum_{n=1}^{\infty} \frac{\log n}{2^n}$

(iii) Show that if $\sum_{n=1}^{\infty} a_n$ converge then $\sum_{n=1}^{\infty} \left(\frac{1+\sin(a_n)}{2}\right)^n$ converges.

Note: Be careful to use general version of ratio test and root test. That is, exist $r < 1$ such that $\frac{a_{n+1}}{a_n} < r$ for all $n > N$, then $\sum a_n$ converge. Note that r must uniform less than 1 *i.e.* $r = 1 - \epsilon$.

(4) [Extra] $\boxed{\text{Raabe Test}}$

Determine whether the series $\sum_{n=1}^{\infty} \frac{1\times 3\times \cdots (2n-1)}{4\times 6\times \cdots (2n+2)}$ converge or not.

(5) $\boxed{\text{Comparison Test}}$

Determine whether the following series converge or not.

(i) $\sum_{n=1}^{\infty} \frac{1}{\log n}$

(ii) [*DIY*] $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n \log n}}$

(iii) $\sum_{n=1}^{\infty} \frac{1}{2^{\log n}}$

(6) $\boxed{\text{Limit Comparison Test}}$

Determine whether the following series converge or not.

(i) $\sum_{n=1}^{\infty} \frac{3n^2+5n}{2^n(n^2+1)}$

(ii) $\sum_{n=1}^{\infty} \left(1 - \cos \frac{1}{n}\right)$

(iii) $\sum_{n=1}^{\infty} \sin \frac{1}{n^2}$

(iv) $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$

(v) Show that if $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ converge then $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$

Hint: Taylor series

(a) $e^x = 1 + x + \frac{x^2}{x!} + \cdots$

(b) $\sin x = x - \frac{x^3}{x!} + \frac{x^5}{5!} + \cdots$

(c) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$

Remark: In my opinion, memorizing special series is more difficult than using Taylor's theorem. **This is remark is for reference only.**

(7) $\boxed{\text{Conditional Converge}}$

Determine whether the following series (absolute) converge or not.

(i) $\sum_{n=1}^{\infty} \frac{n\cos(n\pi)}{n^2+1}$

(ii) $\sum_{n=1}^{\infty} (-1)^n \frac{\log n}{n^2}$

(8) $\boxed{\text{Conditional Converge and Riemann rearrangement theorem}}$

[Extra] Answer the following question?

(i) Is the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$ (absolutely) converge?

(ii) $\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

(iii) $\frac{3}{2}\log 2 = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots$

(iv) Could you rearrange the series such that the value of such series is $2\log 2$.

(v) Could you rearrange the series such that the value of such series is $\log\left(2\sqrt{\frac{p}{q}}\right)$.

(vi) Why?

(9) $\boxed{\text{Exercise}}$

Determine the convergency (absolute convergent/conditional convergent/divergent) of following series.

(i) $\sum_{n=1}^{\infty}(-1)^n \frac{n!}{n^n}$

(ii) $\sum_{n=1}^{\infty}(n^{\frac{1}{n}} - 1)$

(iii) $\sum_{n=1}^{\infty} ne^{-n}$

(iv) $\sum_{n=1}^{\infty} \sinh(\frac{1}{n^2})$

(v) $\sum_{n=9}^{\infty} \frac{1}{n \ln(n) \cdot (\ln(\ln(n)))^2}$

Hint:

(i) **Absolute Convergence.**
By alternative series test and ratio test.

(ii) **Divergence.**
Since $n^{\frac{1}{n}} = e^{\frac{1}{n}\log(n)} \approx 1 + \frac{1}{n}\log(n)$, try to compare with $\frac{1}{n}$.

(iii) **Absolute Convergence.**
By ratio test or root test.

(iv) **Absolute Convergence.**
Since $\sinh(\frac{1}{n^2}) = (e^{\frac{1}{n^2}} - e^{\frac{-1}{n^2}})/2 \approx [(1 + \frac{1}{n^2}) - (1 - \frac{1}{n^2})]/2 = \frac{1}{n^2}$, try to compare with $\frac{1}{n^2}$.

(v) **Absolute Convergence.**
By integral test.