A NOTE OF TAYLOR THEOREM (VERSION 2)

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Remark: Section order is rearranged in this note. Here, I am going to start with Taylor theorem and bring out the problems of Taylor theorem. Then, try to fix these problems by means of tools in your textbook.

(1) $\boxed{\text{The first visit Taylor theorem}}$

Given a function

$$f(x) = \begin{cases} e^{\frac{-1}{x^2}} & \text{, if } x \neq 0 \\ 0 & \text{, if } x = 0 \end{cases},$$

please answer the following questions.

(i) Could you compute Taylor polynomial $\sum_{n=0}^{m} T_n$ about 0?

(ii) Could you find the interval I such that the Taylor series converges on the interval I.?

(iii) Does $\lim_{x\to 0} R_m(x) = 0$ hold for all finite m?

(iv) Does $\lim_{x\to 0} \frac{R_m(x)}{x^m} = 0$ hold for all finite m?

(v) For such interval I mentioned in (ii), does the Taylor polynomial $\sum_{n=0}^{m} T_n$ converge to the origin function $f(x)$ as $m \to \infty$? **Why?**

Hint: The remainder term can be rewrite as $R_m(x) = f(x) - \sum_{n=0}^{m} T_n$.

Hint:

$$\begin{aligned} f(x) &= e^{\frac{-1}{x^2}} \\ f'(x) &= \frac{2}{x^3} e^{\frac{-1}{x^2}} \\ f''(x) &= \left(\frac{4}{x^6} - \frac{6}{x^4} \right) e^{\frac{-1}{x^2}} \\ f^{(n)}(x) &= p_n(1/x) e^{\frac{-1}{x^2}} \end{aligned}$$

where p_n is polynomial with degree n. The last equation can be proven by induction.

Remark: Taylor Theorem does not tell that Taylor polynomial $\sum_{n=0}^{m} T_n$ $\boxed{\text{converges}}$ on which interval I, and whether Taylor polynomial $\boxed{\text{converges}}$ to origin f or not. There are two type of $\boxed{\text{convergence}}$. Hence, if Taylor Theorem is applied, then two things have to be checked:

(i) Determine the interval I such that Taylor polynomial $\sum_{n=0}^{m} T_n$ converges, by ratio test or root test, ...

(ii) Prove the remainder term goes to 0 as $m \to \infty$ on such interval I.

(2) $\boxed{\text{Find the Taylor series}}$

(i) Find the Taylor series for $f(x) = \sin x$ about $x = 0$.

(ii) Find the convergence interval.

(iii) Does Taylor series converge to the $\sin x$ on interval I mentioned in (ii).

(3) $\boxed{\text{Find the Taylor series}}$

(i) Find the Taylor series for $f(x) = \log(1 + x)$ about $x = 0$.

(ii) Find the convergence interval.

(iii) Does Taylor series converge to the $\log(1 + x)$ on interval I mentioned in (ii).

Remark: Let f be smooth and have convergence radius R. Then,

(a) Could you pass $\frac{d}{dx}$ and $\int$ into Σ?

(b) Is convergence radius same? convergence interval *i.e.* end-points?

(4) $\boxed{\text{Find the Taylor series}}$

(i) Find the Taylor series for $f(x) = \int_0^x e^{-t^2} dt$ about $x = 0$. $(x > 0)$

(ii) Find the convergence interval.

(iii) Does Taylor series converge to the f on interval I mentioned in (ii).

(5) $\boxed{\text{Convergence interval}}$

Find the interval of convergence of series

$$\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n .$$

Hint: Show that $\left(1 + \frac{1}{n}\right)^n < e$. This might be applied generalization ratio test so this problem can be dropped out. If you know any other method, please tell me without hesitation.

(6) $\boxed{\text{The second visit the Taylor theorem}}$

Let $f : I \to \mathbf{R}$ be C^m on I with $0 \in I$.

(i) If take $m = 0$, then $f(x) = f(0) + R_0(x)$. Taylor theorem gives $R_0(x) = o(1)$. What is the relation between $f \in C^0$ and $\lim_{x\to 0} R_0(x) = 0$.

(ii) If take $m = 1$, then $f(x) = f(0) + f'(0)x + R_1(x)$. Taylor theorem gives $R_1(x) = o(x)$. What is the relation between $f \in C^1$ and $\lim_{x\to 0} \frac{R_1(x)}{x} = 0$.

(iii) If take $m = 0$ *i.e.* C^0 on interval $[a, b]$ and differentiable on interval (a, b), then $f(x) = f(a) + f'(\xi)(x - a)$, where $x \in (a, b)$ and $\xi \in (a, x)$. What is the relation between above equation and mean value theorem.

Remark: Could you have intuitive sense of Taylor theorem? Taylor theorem tells you if the higher order of Taylor series is known, then not only does the error term goes to zero *i.e.* $\lim_{x\to 0} R_m(x) = 0$ but also is the "speed" of the error term decay with higher order *i.e.* $\lim_{x\to 0} \frac{R_m(x)}{x^m} = 0$. **Hence, checking $\lim_{x\to 0} R_m(x) = 0$ or $\lim_{x\to 0} \frac{R_m(x)}{x^m} = 0$ is not needed, because this is just check whether Taylor theorem is correct or not.**

Let $f : I \to \mathbf{R}$ be C^m on I with $0 \in I$. Then,

$$f(x) = \sum_{n=0}^{m} T_n(x) + R_m(x) ,$$

where $\sum_{n=0}^{m} T_n$ is m-th Taylor polynomial for f about 0 and R_m is Taylor remainder term for f about 0, which satisfy

$$\lim_{x\to 0} \frac{R_m(x)}{x^m} = 0 .$$

That is, $R_m(x) = o(x^m)$.

In addition, let f be C^{m+1}. If Lagrange remainder term is chosen, then

$$f(x) = \sum_{n=0}^{m} T_n(x) + R_m(x) ,$$

and exist ξ between x and 0

$$R_m(x) = \frac{f^{(m+1)}(\xi)}{(m+1)!} x^{m+1} .$$

In fact, Lagrange remainder term only needs $f \in C^m$ and f differentiable everywhere on I. Note that $C^{m+1} \subset C^m$, so Lagrange remainder term also satisfies little o of order m.

Remark: Taylor theorem tells if $f \in C^m$ then remainder term goes to 0 with order m.

Brief proof of Lagrange remainder term: Define $g : t \mapsto \mathbf{R}$ by

$$g(t) = f(t) - \sum_{n=0}^{m} T_n(x) + M(x - c)^{m+1} .$$

Claim $g^{(i)}(c) = 0$ for all $1 \le i \le m$ and $g(x) = f(x) = 0$. By Rolle's theorem, exist ξ between x and c such that $g^{(m+1)}(\xi) = 0$. Then, $M = \frac{f^{(m+1)}(\xi)}{(m+1)!}$ can be solved.

(7) $\boxed{\text{Arithmetic}}$

Find the first 3 terms of Taylor series for $\tan x$ about $x = 0$.

(8) $\boxed{\text{Find the limit}}$

Find the limit

$$\lim_{x \to 0} \frac{e^{\frac{-x^2}{2}} - \cos x}{x^4}.$$

(9) $\boxed{\text{Approximation and error bound}}$

(i) Approximate the following value, (ii) approximate error and (iii) find the error bound.

$$\int_0^{\frac{1}{2}} \frac{1 - \cos x^2}{x} dx$$

(10) $\boxed{\text{Find higher derivative}}$

Let $f(x) = \frac{x - \sin x}{x^3}$. Find the value of $f^{(6)}(0)$.

(11) $\boxed{\text{Converse of Taylor theorem}}$ *remainder term is little o with order m imply $f \in C^m$?*
Given a function

$$f(x) = \begin{cases} e^{\frac{-1}{x^2}} \sin\left(e^{\frac{1}{x^4}}\right) & , \text{ if } x \neq 0 \\ 0 & , \text{ if } x = 0 \end{cases},$$

please answer the following questions.

(i) Find $f(0)$ and $f'(0)$.

(ii) Let $T_1(x) = f(0) + f'(0)x$. Is $\lim\limits_{x \to 0} \frac{f(x) - T_1(x)}{x} = 0$?

(iii) Is $f \in C^1$?

(12) $\boxed{\text{Expand about more points?}}$
Suppose $x_0, x_1, \cdots, x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Then, for $x \in [a, b]$ exist $\xi \in (a, b)$ such that

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0)(x - x_1) \cdots (x - x_n) ,$$

where P is Lagrange interpolating polynomial as you learned in senior high school *i.e.*

$$P(x) = \sum_{k=1}^{n} f(x_k) \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{(x - x_i)}{(x_k - x_i)} .$$

Hint: If $x = x_k$, done!
If $x \neq x_k$, define $g : t \mapsto [a, b]$ by

$$\begin{aligned} g(t) &= f(t) - P(t) - [f(x) - P(x)] \frac{(t - x_0)(t - x_1) \cdots (t - x_n)}{(x - x_0)(x - x_1) \cdots (x - x_n)} \\ &= f(t) - P(t) - [f(x) - P(x)] \prod_{i=0}^{n} \frac{(t - x_i)}{(x - x_i)} \end{aligned} .$$

Claim $g(x_k) = 0$ and $g(x) = 0$. Hence, there are $(n+2)$ points satisfies $g(t) = 0$. By generalize Rolle's theorem, exist $\xi \in (a, b)$ such that $g^{(n+1)}(\xi) = 0$. Note that P is degree n and

$$\frac{d^{n+1}}{dt^{n+1}} \prod_{i=0}^{n} \frac{(t - x_i)}{(x - x_i)} = \frac{(n+1)!}{\prod_{i=0}^{n} (x - x_i)} .$$

Plug into $g^{(n+1)}(\xi) = 0$ and the result follows.

Exercise

(13) $\boxed{\text{Find the limit}}$ *1042 A2 Midterm Problem 3(c)*

Evaluate the following limit:

$$\lim_{x\to 0^+} \frac{\frac{\pi}{2} - \cos^{-1} x - x}{x^3}$$

Hint: I think the following two expansion are most used, but these seem to be not needed to learn, right?

$$\sqrt{1+x} = (1+x)^{\frac{1}{2}} = \binom{\frac{1}{2}}{0} + \binom{\frac{1}{2}}{1} x + \binom{\frac{1}{2}}{2} x^2 + \cdots = 1 + \frac{1}{2}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2 + \cdots$$

$$\frac{1}{1+x} = (1+x)^{-1} = \binom{-1}{0} + \binom{-1}{1} x + \binom{-1}{2} x^2 \cdots = 1 - x + \frac{(-1)(-2)}{2!}x^2 + \cdots$$

(14) $\boxed{\text{Find higher derivative}}$ *1012 A2 Midterm Problem 4(b)*

Let $f(x) = \log\left(\sqrt{\frac{1+x^2}{1-x^2}}\right)$. Find $f^{(10)}(0)$.

(15) $\boxed{\text{Approximation}}$ *1062 A1 Midterm Problem 3(b)*

Let $f(x) = \int_0^x \log\left(1 + \frac{t^2}{2}\right)\, dt$. Estimate $f(0.1)$ up to an error 10^{-7}.