CALCULUS TA SESSION DECEMBER 19 (VERSION 1)

(1) Definition

Let

$$f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}.$$

Answer the following questions.

(i) Is f continuous?
(ii) Compute $f_x(x,y)$ and $f_y(x,y)$ where $(x,y) \neq 0$.
(iii) Compute $f_x(0,0)$ and $f_y(0,0)$.
(iv) Determine $f_{xy}(0,0)$ and $f_{yx}(0,0)$.
(v) Does directional derivative $D_u f$ exist for all unit vector u.
(vi) Compute $\nabla f \cdot u$
(vii) Is f differentiable?

Remark: What is the concept of differentiable in higher dimension. In 1-dim, if a line could be found to approximate a curve, i.e. exist m such that $f(x+h) = f(x)+mh+o(h)$, then we say f is differentiable. In particular, $m = f'(x)$. Similarly, in 2-dim, if a plane could be found to approximate surface, *i.e.* $f(x+h, y+k) = f(x,y)+hf_x(x,y)+kf_y(x,y)+o(\sqrt{h^2+k^2})$, then we say f is differentiable. In fact, $L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) = 0$ is tangent plane at (a,b).

(2) [*DIY*] Let

$$f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}.$$

Answer the following questions.

(i) Is f continuous? Along $y = x^2$? Along $y = mx$?

(ii) Compute $f_x(0,0)$ and $f_y(0,0)$.

(iii) Does directional derivative $D_u f$ exist for all unit vector u.

(3) Gradient

Let $f(x,y,z) = e^{xy} \log z$. Find the directional derivatives of f at $P(1,0,e)$ in the following directions:

(i) In the direction in which f increases most rapidly at P.

(ii) In the directions parallel to the line in which the planes $x+y-z = 2$ and $4x-y-z = 1$ intersect.

Remark:

(a) Could you show the fact that the gradient vector $\nabla f(x_0, y_0)$ is perpendicular to the level curve $\{(x,y) : f(x,y) = k\}$ at point (x_0, y_0).

(b) Could you show the fact that the maximum value of $D_u f$ is $\|\nabla f\|$ which occur in direction ∇f. That is, maximum rate of change of the function f.

(4) Direction Derivative

Let $f(x, y, z) = (x^2 + z^2)\sin\left(\frac{\pi x y}{2}\right) + yz^2$ and a point $(1, 1, -1)$ called p.

(i) Find the gradient of f at p.

(ii) Find the approximate value of $f(0.98, 1.02, -0.97)$.

(iii) Find the plane tangent to the level surface through p defined by $f(p) = 3$

(iv) If a bird flies through p directly to the point $(2, -1, 1)$ with speed 5, what is the rate of change of f as seen by the bird at p? **Why the norm u always equals to 1?**

(5) Second derivative test Find the local maximum and minimum values and saddle point(s) of the function

$$f(x, y) = x^3 - y^3 + 3x^2 + 3y^2 - 9x$$

Remark: Recall Taylor Theorem

$$f(x + h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + o(h^2)$$

Moreover,

$$f(p + h) = f(p) + \nabla(p) \cdot h + \frac{1}{2}h^T Hess(p)h + o(h^2)$$

(6) Implicity function theorem Suppose that x, y, z satisfy the relation $x^2+2y^2+3z^2+xy-z=9$. Find $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y^2}$.

(7) Change order of second derivative

(i) Let a function

$$f(x,y)=\begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$$

Find the value of $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$. **Why?** Is f, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ continuous near $(0,0)$?

(ii) [*DIY*] Let a function

$$f(x,y)=\begin{cases} \frac{x^2y^2}{x^2+y^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$$

Is $f\in C^2$ near $(0,0)$? Is $\frac{\partial^2 f}{\partial x \partial y}$ continuous near $(0,0)$? Is $\frac{\partial^2 f}{\partial x \partial y}=\frac{\partial^2 f}{\partial y \partial x}$?

(8) Show that $f(x,y) = \log(x^2+y^2) + \tan^{-1}\frac{y}{x}$ satisfy $f_{xx} + f_{yy} = 0$, which is called harmonic function. ($z = x + iy \in \mathbb{C}$)

(9) Show that a function form that $z = f(x+vt) + f(x-vt)$ is a solution of $z_{tt} = v^2 z_{xx}$, which is called wave equation. Image that a wave propagate to both side.