CALCULUS TA SESSION DECEMBER 26 (VERSION 1)

(1) [*DIY*] Without constraint: Second derivative test

Let n points $(x_1, y_1), \cdots, (x_n, y_n)$ with $x_i \neq x_j$ if $i \neq j$. Find $y = ax + b$ satisfied $f(a, b) = \sum_{i=1}^{n}(ax_i + b - y_i)^2$ has minimum.

(2) With constraint: Lagrange multipliers

Find the minimum value taken on by the function $f(x, y) = \frac{x^2}{2} + (y - 1)^2$ on the hyperbola $x^2 - y^2 = 1$.

Remark: Recall Taylor Theorem

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + o(h^2)$$

For instance,

(a) if $f'(x) = 0$ and $f''(x) > 0$, then exist neighborhood of x, ($i.e.$ $\exists \delta > 0$ s.t. $|h| < \delta$), such that $f(x+h) - f(x) = \frac{f''(x)}{2}h^2 + o(h^2) > 0$. (prove it by $\lim\limits_{h\to 0} \frac{o(h^2)}{h^2} = 0$) Hence, f has local minimum at x.

(b) if $f'(x) = 0$ and $f''(x) < 0$, then exist neighborhood of x, ($i.e.$ $\exists \delta > 0$ s.t. $|h| < \delta$), such that $f(x+h) - f(x) = \frac{f''(x)}{2}h^2 + o(h^2) < 0$. (prove it by $\lim\limits_{h\to 0} \frac{o(h^2)}{h^2} = 0$) Hence, f has local maximum at x.

Moreover, write ∇f and h are column vector 2×1,

$$f(p+h) = f(p) + \nabla(p) \cdot h + \frac{1}{2}h^T Hess(p)h + o(\|h\|^2)$$

Similarly,

(a) if $\nabla f(p) = 0$ and $h^T Hess(p)h > 0$, then exist neighborhood of x, ($i.e.$ $\exists \delta > 0$ s.t. $\|h\| < \delta$), such that

$$f(x+h) - f(x) = \frac{1}{2}h^T Hess(p)h + o(\|h\|^2) > 0\,,$$

(prove it by $\lim\limits_{h\to 0} \frac{o(\|h\|^2)}{\|h\|^2} = 0$.) Hence, f has local minimum at x.

(b) if $\nabla f(p) = 0$ and $h^T Hess(p)h < 0$, then exist neighborhood of x, ($i.e.$ $\exists \delta > 0$ s.t. $\|h\| < \delta$), such that

$$f(p+h) - f(p) = \frac{1}{2}h^T Hess(p)h + o(\|h\|^2) < 0\,,$$

(prove it by $\lim\limits_{h\to 0} \frac{o(\|h\|^2)}{\|h\|^2} = 0$.) Hence, f has local maximum at x.

Therefore, **how to determine the sign of $h^T Hess(p)h$?** The proof was shown in your class. The following is the result which has to be remembered.

(a) If $H_{11} = f_{xx}(p) > 0$ and $\det Hess(p) > 0$ then $h^T Hess(p)h > 0$, which impies f has local minimum at p. In fact, we called $Hess(p)$ positive definite.

(b) If $H_{11} = f_{xx}(p) < 0$ and $\det Hess(p) > 0$ then $h^T Hess(p)h < 0$, which impies f has local maximum at p. In fact, we called $Hess(p)$ negative definite.

All cases of the sign of $h^T Hess(p)h < 0$ can be found in your textbook.

(3) Exchange integral

Evaluate

(i)

$$\int_0^{\log 10} \int_{e^x}^{10} \frac{1}{\log y} dydx$$

(ii)

$$\int_0^{\pi} \int_x^{\pi} \frac{\sin y}{y} dydx$$

Hint[1]: Compare the following with triple integral

(a) Section Method

$$\iint f dxdy = \int \left[\int_{sect(y)} f dx \right] dy$$

(b) Projection Method

$$\iint f dxdy = \int_{proj2(y)} \left[\int f dx \right] dy$$

(4) [*DIY*]

$$\iint_\Omega xy^2 dA, \quad \Omega = \left\{ (x, y) : x^2 \leq y \text{ and } y^2 \leq x \right\}$$

Hint: 1062 B Quiz 3 Problem 2

Answer: 3/56

[1]This is my experience. For reference only.

(5) Coordinate change: Polar coordinate

Evaluate

$$\iint_D \frac{\log(x^2+y^2)}{\sqrt{x^2+y^2}}\,,$$

where $D = \{(x,y) : 1 \le x^2+y^2 \le e\}$.

(6) [*DIY 50%*] Coordinate change: Polar coordinate

Evaluate

$$\iint_D \sqrt{x^2+y^2}dA\,,$$

where $D = \{(x,y) : (x-1)^2+y^2 \le 1\}$.

Hint: 32/9

(7) [*Extra?*] Coordinate change: Polar coordinate

Evaluate

$$\iint_D \frac{1}{\sqrt{x^2+y^2}}dA\,,$$

where D is a region bounded by $r = 1-\cos\theta$, $y = x$ and $y = -x$ with $y \ge 0$.

Hint: 1052 B Midterm Problem 7

Answer: $\pi/2$

(8) Coordinate change: linear

Evaluate

$$\iint_D e^{\frac{x-y}{x+y}}\, dA\,,$$

where the region D is enclosed by $x = 0$, $y = 0$ and $x + y = 2$.

(9) Coordinate change: nonlinear

Evaluate

$$\iint_D e^{xy} dA\,,$$

where the region D is enclosed by $y = 1$, $y = 3$, $xy = 1$ and $xy = 4$.

(10) [*Extra?*] Application

Find the volume of solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$.

Hint: $\frac{2\pi}{3}\left(1 - \frac{\sqrt{2}}{2}\right)$

(11) [*Extra?*] Triple integral

Evaluate

$$\iiint_E z dV \,,$$

where E is solid region bounded by $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

Hint: 1/24

Hint: Compare the following with double integral

(a) Section Method

$$\iiint f dz dy dx = \int \left[\iint_{sect(x)} f dz dy \right] dx = \int \left[\int_{sect(x)} \left[\int_{sect(xy)} f dx \right] dy \right] dz$$

(b) Projection Method

$$\iiint f dz dy dx = \iint_{proj2(xy)} \left[\int f dz \right] dy dx = \int_{proj2(x)} \left[\int_{proj2(xy)} \left[\int f dz \right] dy \right] dx$$

(12) [*Extra*] Fubini Theorem

Consider the function $f(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}$ for $(x,y) \in [0,1]\times[0,1]$ and compute the following integrals:

Hint: $\frac{\partial}{\partial x}\frac{-x}{x^2+y^2} = \frac{\partial}{\partial y}\frac{y}{x^2+y^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$

(a) $u(y) = \int_0^1 f(x,y)\ dx$, if $y = 0$ and $0 < y \leq 1$

(b) $\int_0^1 \int_0^1 f(x,y)\ dxdy$

(c) $v(x) = \int_0^1 f(x,y)\ dy$, if $x = 0$ and $0 < x \leq 1$

(d) $\int_0^1 \int_0^1 f(x,y)\ dydx$

(e) $\iint_{[0,1]\times[0,1]} \max\{f(x,y),0\}\ dA$

(f) $\iint_{[0,1]\times[0,1]} \max\{-f(x,y),0\}\ dA$

(g) $\iint_{[0,1]\times[0,1]} |f(x,y)|\ dA$

(h) $\iint_{S_\epsilon} f(x,y)\ dA$, where $S_\epsilon = [0,1]\times[0,1] \setminus [0,\epsilon]\times[0,\epsilon]$

(i) $\iint_{[0,1]\times[0,1]} f(x,y)\ dA$

(j) **Why?** Does (a)(c) exist? Does (g) exist?