今天上課講義

Include slant asymp

12 P.p. 14-15

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(1) $\boxed{\text{Differentiable and Continuous}}$

$$f(x) = \begin{cases} \frac{\sin^2(ax)}{x}, & x > 0 \\ |2x+1| - |2x-1| + b\cos x, & x \le 0 \end{cases}$$

(a) For what values of a and b will $f(x)$ be continuous at $x = 0$?

(b) For what values of a and b will $f(x)$ be differentiable at $x = 0$?

(a) by def: $f(0) = |1| - |-1| + b\cos(0) = b$

First $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} |2x+1| - |2x-1| + b\cos x = b$

$\Rightarrow b = b$

Second $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{\sin^2(ax)}{x} = \lim_{x \to 0^-} \frac{\sin^2 ax}{a^2x^2} a^2 x = 0$

for all a

Hence $b = 0$

Therefore for $a \in \mathbb{R}$, $b = 0$

(b) Since diff $\Rightarrow$ cont, $\boxed{b = 0}$

$$f(x) = \begin{cases} \dfrac{\sin^2(ax)}{x} & x > 0 \\ |2x+1| - |2x-1| & x \le 0 \end{cases}$$

By def $f'(0)$ exist,

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} \qquad \text{if } x<0 \;\; 2x-1<0$$

$$\lim_{x\to 0^-} \frac{|2x+1|-|2x-1|}{x-0} = \lim \frac{(2x+1)-[-(2x-1)]}{x}$$

$$= \lim \frac{4x}{x} = 4$$

$$\lim_{x\to 0^+} \frac{\frac{\sin^2(ax)}{x} - 0}{x} = \lim \boxed{\frac{\sin^2(ax)}{a^2x^2}}\, a^2$$

$$= \begin{cases} 0 & \text{if } a = 0 \\ a^2 & \text{if } a \neq 0 \end{cases} = a^2$$

If f' exist $\Leftrightarrow a^2 = 4 \Rightarrow a = \pm 2$

$$\begin{cases} a = \pm 2 \\ b = 0 \end{cases}$$

(3) $\boxed{\text{Find limit}}$

Determine α, β such that

$$\lim_{x\to\infty} \sqrt{4x^2-3x+2} - (\alpha x + \beta) = 0$$

$\alpha = -2 \quad \beta = \frac{3}{4}$

For convenience,

$$\sqrt{4x^2-3x+2} - (\alpha x+\beta) \times \frac{\sqrt{4x^2-3x+2}+(\alpha x+\beta)}{}$$

$$= \frac{4x^2-3x+2-(\alpha x+\beta)^2}{\sqrt{4x^2-3x+2}+(\alpha x+\beta)}$$

$$= \frac{x^2\left[(4-\alpha^2) - (3+2\alpha\beta)\frac{1}{x} + (2-\beta)\frac{1}{x^2}\right]}{|x|\sqrt{4-\frac{3}{x}+\frac{2}{x^2}} + x\left(\alpha + \frac{\beta}{x}\right)}$$

As $x \to -\infty \Rightarrow |x| = -x$

$$\lim_{x\to-\infty} \frac{x^2\left[(4-\alpha^2) - (3+2\alpha\beta)\frac{1}{x} + (2-\beta)\frac{1}{x^2}\right]}{\cancel{|x|}\sqrt{4-\frac{3}{x}+\frac{2}{x^2}} + x\left(\alpha+\frac{\beta}{x}\right)}$$

$-x$

deg = 1

exist

deg = 1

$$= \frac{-(3+2\alpha\beta)}{-2+\alpha} = 0$$

if $4-\alpha^2=0 \Rightarrow \alpha = \pm 2$

If $\alpha = 2 \Rightarrow -\infty$ X

$\alpha = -2$

$\Rightarrow \dfrac{-(3+4\beta)}{-4} = 0 \quad \Rightarrow \beta = \dfrac{3}{4}$

(4) $\boxed{\text{Find limit}}$

Compute the following limit

$$\lim_{x\to\infty} x(\sqrt{x^6 - 3x^5 + 1} - x^3)\tan\left(\frac{1}{x^3}\right)$$

$$-\frac{3}{2}$$

$$\infty(\infty - \infty)\cdot 0$$

$$x\left(\sqrt{x^6 - 3x^5 + 1} - x^3\right) = \frac{\sqrt{x^6 - 3x^5 + 1} + x^3}{-1}$$

$$= \frac{x(\cancel{x^6} - 3x^5 + 1 - \cancel{x^6})}{|x|^3\sqrt{1 - \frac{3}{x} + \frac{1}{x^6}} + x^3} \quad -x^5$$

$$\lim_{x\to\infty} \frac{x^2\left(-3+\frac{6}{x^5}\right)}{x^3\left[\sqrt{1-\frac{3}{x}+\frac{1}{x^6}}+1\right]} \cdot \frac{x\sin\left(\frac{1}{x^3}\right)}{\cos\left(\frac{1}{x^3}\right)} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$

0

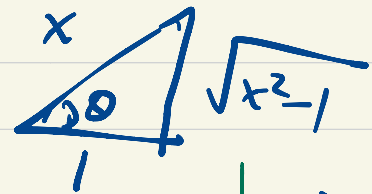
$x>0$

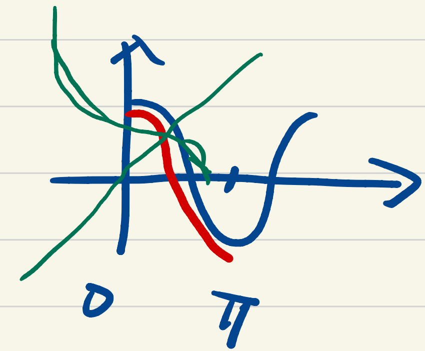
$$= \frac{-3}{2}$$

(5) $\boxed{\text{Find inverse function}}$

Find $\sin \sec^{-1} x$ for $x \leq -1$.

$\cos^{-1} : [-1, 1] \to [0, \pi]$





$\theta = \sec^{-1} x \qquad 0 \geq \cos\theta \geq -1$

$\Rightarrow \sec\theta = x \qquad \Rightarrow \boxed{\cos^{-1}\frac{1}{x} = \theta}$

$\Rightarrow \cos\theta = \frac{1}{x} \qquad \Rightarrow \theta \in \left(\frac{\pi}{2}, \pi\right] \quad \in \mathrm{II}$

$\sin\theta = \pm\sqrt{1 - \frac{1}{x^2}}$

Since $x \leq -1$

$\Rightarrow 0 \geq \frac{1}{x} \geq -1$

$\sin^2\theta = 1 - \cos^2\theta$

$= 1 - \frac{1}{x^2}$

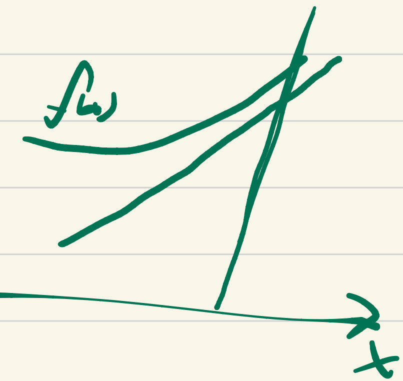
(7) $\boxed{\text{Find slant asymptotes}}$ *1041 A1 Midterm Problem 9*

Let $f(x) = (x^3 + x^2)^{1/3}$. Find all asymptotes of $f(x)$.

Hint: $y = x + \frac{1}{3}$

check vertical (horizontal) asymp !

↳ $m = \infty$



Ansatz $L(x) = mx + b$,

$\exists\, a$ s.t. $\lim_{x \to a} f(x) = \pm\infty$

First step, find m

$$\lim_{x \to \infty} \frac{f(x)}{x} = \lim \frac{(x^3 + x^2)^{\frac{1}{3}}}{x} = 1$$

$mx = x$

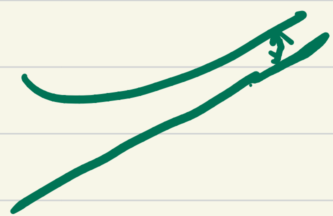
$$b = \lim_{x \to \infty} \left[f(x) - x \right]$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

$$= \lim_{x \to \infty} \underbrace{\left[(x^3+x^2)^{\frac{1}{3}} - x \right]}_{} \times \frac{(x^3+x^2)^{\frac{2}{3}} + x(x^3+x^2)^{\frac{1}{3}} + x^2}{〃}$$

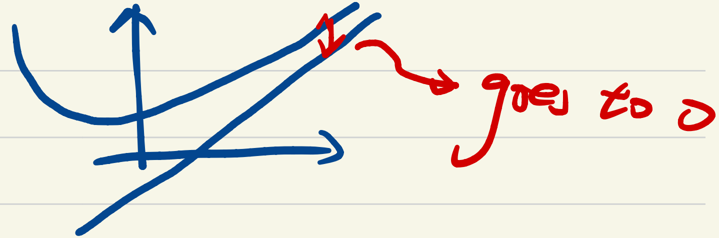
$$= \lim \frac{1}{〃} = \frac{1}{3}$$

$$y = x + \frac{1}{3}$$



Supplement of Slant Asymptotes

If $\lim\limits_{x\to\pm\infty}\left[f(x)-(mx+b)\right]=0$



$\Rightarrow$ the line $L(x)=mx+b$ is the asymp to graph $y=f(x)$

Note that if $m=0 \Rightarrow$ horizontal

* Note that if $\lim\limits_{x\to\pm\infty}\left[f(x)-(mx+b)\right]$ doesn't exist which doesn't means there is no asymp (maybe vertical, i.e. $m=\infty$)

How to find m and b to $L(x) = mx + b$

First, consider

$$\lim_{x \to \pm\infty} \frac{f(x) - (mx+b)}{x} = \lim_{x \to \pm\infty} \frac{f(x)}{x} - \left(m + \frac{b}{x}\right)$$

$$\Rightarrow m = \lim_{x \to \pm\infty} \frac{f(x)}{x}$$

Second, by definition of asymp.

$$\lim_{x \to \pm\infty} \left[f(x) - (mx+b)\right] \Rightarrow b = \lim_{x \to \pm\infty} \left[f(x) - mx\right]$$