CALCULUS TA SESSION FOR GROUP 1 OCTOBER 21

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(1) Differentiable and Continuous

Let function defined as follows

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}\,, & \text{if } x > 0 \\ ax + b\,, & \text{if } x \le 0\,. \end{cases}$$

(a) Find $f'(x)$ for $x \neq 0$.

(b) What value should be assign to a, $b \in \mathbb{R}$ such that f is differentiable at $x = 0$.

(c) From (b), is f' continuous at $x = 0$?

(2) Mean Value Theorem `1051 A1 Midterm Problem 6`

Suppose that f is a differentiable function. If $f'(a) > 0$ and $f'(b) < 0$, explain that there exists $c \in (a, b)$ such that $f'(c) = 0$.

Remark: The function f is differentiable at c, which doesn't imply f' continuous at c. That is, differentiable doesn't imply C^1.

(3) Differentiable of inverse function `1041 A1 Midterm Problem 3`

Let $f(x)$ be a twice differentiable one-to-one function. Suppose that $f(2) = 1$, $f'(2) = 3$, $f''(2) = e$. Find the following value

$$\frac{d}{dx} f^{-1}(1) \quad \text{and} \quad \frac{d^2}{dx^2} f^{-1}(1)$$

(4) Differentiable of inverse function `1071 A1 Midterm Problem 2`

Use the implicit differentiation theorem to derive

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}\,.$$

Remark: $\left(\frac{1}{x}\right)'$, $(e^x)'$, $(2^x)'$, $(\log(x))'$.

(5) Implicity differentiation `108 A1 Midterm Problem 3`

Suppose function g has the following property

$$g(\sin 3x) = 2(g(x) + x)$$

for any real number x and g is differentiable at $x = 0$. Find $g(0)$ and $g'(0)$.

(6) **This is important concept to extend 1-dimensional differentiable sense to higher dimension by approximation** *i.e.* **little** o**.**

Let the function $f : [a, b] \to \mathbb{R}$ be continuous. Then, f is differentiable if and only if the following statement following:

there exists $m \in \mathbb{R}$ such that $f(c+h) = f(c)+mh+o(h)$ for h is small sufficient which satisfied $\lim\limits_{h\to 0} \frac{o(h)}{h} = 0$. Moreover, m is unique equal to $f'(c)$.

Solution:

$\Leftarrow$) Since the definition of the function o,

$$\lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h\to 0} \frac{f(c) + mh + o(h) - f(c)}{h} = m + \lim_{h\to 0} \frac{o(h)}{h} = m\,.$$

Thus, above limit exist which implies f is differentiable and $f'(c) = m$.

$\Rightarrow$) Since $f'(c)$ exists so $\lim\limits_{h\to 0} \frac{f(c+h)-f(c)}{h}$ exists, called m. Define a function $o : I \to \mathbb{R}$ by $o(h) = f(c+h) - f(c) - mh$. Now it's sufficient to prove $\lim\limits_{h\to 0} \frac{o(h)}{h} = 0$, which do it by yourselves. [*incomplete*]

(7) Limit

Find the limit

$$\lim_{x\to\infty}\left(1+\frac{3}{x}+\frac{5}{x^2}\right)^x$$

Hint: Consider

$$\log\left(1+\frac{3}{x}+\frac{5}{x^2}\right)^x = \frac{\log\left(1+\frac{3}{x}+\frac{5}{x^2}\right)}{\frac{1}{x}}.$$

Second, let $h=\frac{1}{x}\geq 0$, $f(h)=\log(1+3h+5h^2)$. Then, $f(0)=0$ and $f'(0)=3$. Hence,

$$\lim_{x\to\infty}\log\left(1+\frac{3}{x}+\frac{5}{x^2}\right)^x = 3$$

(8) Limit

Find the limit

$$\lim_{x\to\infty}\left(\frac{a+x}{a-x}\right)^{\frac{1}{x}}$$

for $a>0$.

Hint: Consider

$$\left(\frac{a+x}{a-x}\right)^{1/x} = \frac{(1+\frac{x}{a})^{\frac{1}{x}}}{(1-\frac{x}{a})^{\frac{1}{x}}} = \frac{\left((1+\frac{x}{a})^{\frac{a}{x}}\right)^{1/a}}{\left((1-\frac{x}{a})^{\frac{-a}{x}}\right)^{-1/a}}$$

(9) Tangent line `1081 AM1 Midterm Problem 4`

Find the equation of the tangent line to the curve satisfying $x^{2/3} + y^{2/3} + y = 6$ at $(8, 1)$.

(10) Tangent line

Given astroid has an equation of the form $x^{2/3} + y^{2/3} = a$ where $a > 0$. Find $\frac{dy}{dx}$ and show that the length of the portion of any tangent line to the astroid cut off by the coordinate axes is constant.

Hint: It's sufficient to compute $y' = -x^{-1/3}y^{1/3}$. Assume the tangent line tangent to (x_0, y_0), so

$$y = -x^{-1/3}y^{1/3}(x - x_0) + y_0 \, .$$

We have x-intercept is $(x_0 + x_0^{1/3}y_0^{2/3}, 0)$ and y-intercept is $(0, y_0 + x_0^{2/3}y_0^{1/3})$. Just calculus the distance between two points. The answer is $a^{3/2}$.