CALCULUS TA SESSION FOR GROUP 1 NOVEMBER 4

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(1) Mean Value Theorem `1051 A1 Midterm Problem 6`
Suppose that f is a differentiable function. If $f'(a) > 0$ and $f'(b) < 0$, explain that there exists $c \in (a, b)$ such that $f'(c) = 0$.

(2) Mean Value Theorem `1061 B Midterm Problem 2`
Show that for $y > x \geq 0$, then

$$\tan^{-1} y - \tan^{-1} x \leq (y - x) .$$

Hint: $(\tan^{-1} x)' = \frac{1}{1+x^2}$

(3) Linear approximation `1091 AM Midterm Problem 4`

Suppose that near the point $(3, 8)$, a function $y = f(x)$ is defined as

$$3y^{2/3} + xy = 36\,.$$

(a) Compute $\frac{df}{dx}$ at $(3, 8)$. Remark that compare to $f'(3)$.

(b) Use the linear approximation to estimate $f(3.01)$.

(c) Use second derivative to determine whether the estimation from (b) is larger or smaller than real $f(3.01)$.

(4) **This is important concept to extend 1-dimensional differentiable sense to higher dimension by approximation** *i.e.* **little** o**.**

Let the function $f : [a, b] \to \mathbb{R}$ be continuous. Then, f is differentiable if and only if the following statement following:

there exists $m \in \mathbb{R}$ such that $f(c+h) = f(c)+mh+o(h)$ for h is small sufficient which satisfied $\lim\limits_{h\to 0} \frac{o(h)}{h} = 0$. Moreover, m is unique equal to $f'(c)$.

Solution:

$\Leftarrow$) Since the definition of the function o,

$$\lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h\to 0} \frac{f(c) + mh + o(h) - f(c)}{h} = m + \lim_{h\to 0} \frac{o(h)}{h} = m\,.$$

Thus, above limit exist which implies f is differentiable and $f'(c) = m$.

$\Rightarrow$) Since $f'(c)$ exists so $\lim\limits_{h\to 0} \frac{f(c+h)-f(c)}{h}$ exists, called m. Define a function $o : I \to \mathbb{R}$ by $o(h) = f(c+h) - f(c) - mh$. Now it's sufficient to prove $\lim\limits_{h\to 0} \frac{o(h)}{h} = 0$, which do it by yourselves. [*incomplete*]

(5) L'Hopital Rule

Let

$$f(x) = x + \cos x \sin x$$
$$g(x) = e^{\sin x}(x + \cos x \sin x)$$

Calculate the following limits

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} \qquad \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

Why? Does L'Hopital rule fail?

Hint:

$$\begin{cases} f'(x) & = 2\cos^2 x \\ g'(x) & = e^{\sin x}\cos x(x + \cos x \sin x + 2\cos x) \end{cases}$$

Hence, $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1}{e^{\sin x}}$ and $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = 0$

(6) Mean Value Theorem `1041 B Midterm Problem 6`

Show that there is only one intersection of the graphs $y = 1 - x$ and $y = \cos x$.

(7) Let $y = f(x) = \frac{x|x+1|}{x+2}$.

(a) Find the local maximum and local minimum values on $(-\infty, \infty)$.

(b) Find the inflection points.

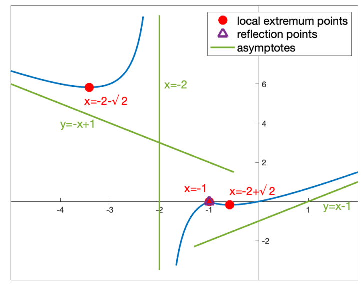
(c) Find asymptotes of the curve

(d) Sketch the graph of $y = f(x)$.

Hint:

Rewrite the function

$$f(x) = \begin{cases} -\frac{x(x+1)}{x+2} & \text{if } x \leq -1 \text{ and } x \neq -2 \\ \frac{x(x+1)}{x+2} & \text{if } x > -1 \end{cases} .$$



Then,

$$f'(x) = \begin{cases} -(\frac{x(x+1)}{x+2})' = -\frac{x^2+4x+2}{(x+2)^2} & \text{if } x < -1 \text{ and } x \neq -2 \\ (\frac{x(x+1)}{x+2})' = \frac{x^2+4x+2}{(x+2)^2} & \text{if } x > -1 \end{cases}$$

Moreover, f is not differentiable at $x = -1$ by

$$\lim_{x \to -1^-} \frac{f(x) - f(-1)}{x+1} = -\frac{1}{3} \neq \frac{1}{3} = \lim_{x \to -1^+} \frac{f(x) - f(-1)}{x+1} .$$

And,

$$f''(x) = \begin{cases} \frac{-4}{(x+2)^3} & \text{if } x < -1 \text{ and } x \neq -2 \\ \frac{4}{(x+2)^3} & \text{if } x > -1 \end{cases}$$

Note that $f''(x)$ is not defined at $x = -2, -1$