Calculus II TA Session

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TA: Sing-Yuan Yeh

1. **(Separable Equations)** 1111 (01-05) Final Problem 5

In country A, a booster vaccine has been invented to conquer a mutated virus X. Let x (in thousands) be the total number of people in country A at time t (in months) and y be the those who has received the booster shot at time t. It is known that $0 < x < 100$ and x, y satisfies the following equations.

$$(1) \quad \frac{dx}{dt} = 0.2x\left(1 - \frac{x}{100}\right), \qquad (2) \frac{dy}{dt} = 0.4y\left(1 - \frac{y}{x}\right)$$

It is known that $x(0) = 10$ and $y(0) = 1$.

(a) By solving (1), find x in terms of t. Express your answer in the form $x(t) = \frac{100}{f_1(t)}$.

(b) By letting $u = \frac{1}{y}$ in (2), find y in terms of t. Express your answer in the form $y(t) = \frac{100}{f_2(t)}$.

(c) Hence, determine how long will it take for 80% of the population to have received the booster vaccine.

2. **(ODE)** 1071 A1 Final Problem 4

   1. Show that $y = 0$ is an orthogonal trajectory of the family of curves $x^2 + \frac{y^2}{k} = 1$, where $k > 0$ is an arbitrary constant. Find the orthogonal trajectories of the same family of curves when $y \neq 0$.
   2. Find $u(t)$ that satisfies the ordinary differential equation

   $$u'(t) + \ln(t)u(t) = e^{-t\ln(t)}, \quad t > 0$$

   and the condition

   $$\lim_{t \to 0^+} u(t) = 2$$

3. **(ODE)** 1081 A1 Final Problem 6

(a) Find the orthogonal trajectories of the family of curves $y = C \tan x$, where C is an arbitrary constant.

(b) Solve the differential equation $(\cos x) \cdot y' + (\sin x) \cdot y = \tan x, y(0) = 1$.

4. **(Improper integral)** 1091 (01-05) Final Problem 1

Show that $\int_0^{\frac{\pi}{2}} e^x \csc x dx$ diverges to infinity by the comparison test.

Solution: Use the Taylor expansion to find the test function.

(i) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

(ii) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

(iii) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

(iv) $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$ for $|x| < 1$

(v) $\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$ for $|x| < 1$

5. **(Improper integral)** 1121 (11-14) Quiz 2 & make-up Problem 3
Test the convergence of

$$\int_1^e \frac{dx}{\ln x} \qquad \text{and} \qquad \int_0^1 \frac{dx}{\sqrt{\sin x}}$$

6. **(Improper integral)** 1111 (01-05) Final Problem 3

   (a) Determine whether $\int_1^\infty \frac{\sin x}{x}\,\mathrm{d}x$ is convergent or divergent.

   (b) (option) Show the Riemann sum on $[\epsilon, 1]$ of $\int_0^1 \frac{\sin(1/x)}{x}\,\mathrm{d}x$ will depend on ϵ which is closed to 0.

> **Solution:** Let $f(x) = \frac{\sin(1/x)}{x}$ and partition $\mathcal{P} = \{0 < \epsilon < x_1 < x_2 < \cdots < x_n = 1\}$
>
> $$\epsilon \cdot f(\epsilon) + (x_1 - \epsilon) f(x_1) + \sum_{k=2}^{n} f(x_k)(x_k - x_{k-1}) = \epsilon (f(\epsilon) - f(x_1)) + \sum_{k=1}^{n} f(x_k)(x_k - x_{k-1}).$$

7. **(FTOC)** 1051 A1 Final Problem 8 Let $f(x)$ be a differentiable and increasing function on $[a, b]$, where $a > 0$. Find a horizontal line $y = L$ that will minimize the function $F(L) = \int_a^b x|f(x) - L|\mathrm{d}x = \int_a^{f^{-1}(L)} x(L - f(x))\mathrm{d}x + \int_{f^{-1}(L)}^b x(f(x) - L)\mathrm{d}x$.