Calculus Ⅲ TA Session

February 22, 2024

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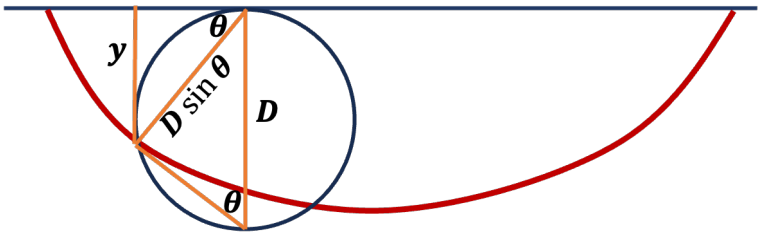
1. Brachistochrone shortest-time path Given the cycloid curve $\{(x, y) : x = \theta - \sin\theta, y = -1 + \cos\theta \text{ and } 0 \le \theta \le 2\pi\}$, find the following value
   (a) the area of between the cycloid curve and x-axis.
   (b) the arc-length of the cycloid curve.
   (c) [Extra] Could you find the value of $\frac{\sin\theta}{\sqrt{y}}$?
   (d) [Extra] Is it similar to Snell's law?
   (e) [Extra] Could you find a path in which the particle takes minimal time to move between two places by gravity.

Solution:

(c) Let's consider the following figure

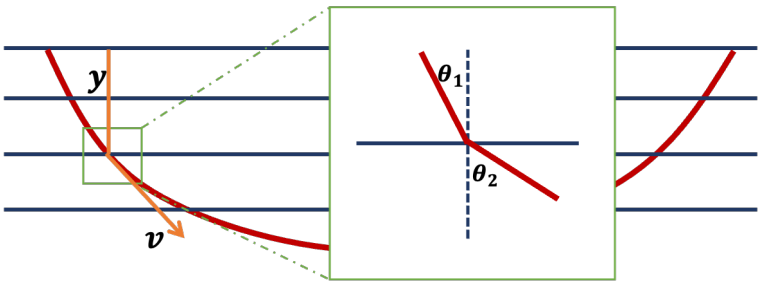


We know $y = D\sin^2\theta$. Hence,

$$\frac{\sin(\theta)}{\sqrt{y}} = \frac{1}{\sqrt{D}}$$

is a constant. In this case diameter $D = 1$.

(d) Let's consider the following figure



By the conservation of energy, we have $\frac{1}{2}mv^2 = mgy$. Then, $v = \sqrt{2gy}$. By Snell's law,

$$\frac{\sin(\theta_1)}{v_1} = \frac{\sin(\theta_2)}{v_2}$$

Substitute $v = \sqrt{2gy}$ into above equation,

$$\frac{\sin(\theta_1)}{\sqrt{y_1}} = \frac{\sin(\theta_2)}{\sqrt{y_2}}$$

According to (c), we know above quantity is a constance.

2. **(Parametric Equations)** 1111 (11-14) Midterm Problem 1 A curve C is defined by the parametric equations $x = 2t - \pi\sin(t), y = 2 - \pi\cos(t)$, where $-\pi < t < \pi$.

   (a) Find $\frac{dy}{dx}$.

   (b) Show that C has two tangents at the point $(x, y) = (0, 2)$ and find the equations of these tangent lines.

   (c) Find $\frac{d^2y}{dx^2}$. Is C concave upward or downward near $t = \frac{\pi}{3}$?

   (d) Find the area of region which is enclosed by the curve $C, x = 2\pi$ and $y = 2$.