Calculus III TA Session

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Remark. Let $k \le n$ and $f : \mathbb{R}^n \to \mathbb{R}^1$ and $g_i : \mathbb{R}^n \to \mathbb{R}^1$ for $1 \le i \le k$. Consider $\mathbf{a} \in \mathbb{R}^n$. The hypothesis is that

(i) f and g_i are C^1;

(ii) $\{\nabla g_i|_{\mathbf{a}}\}_{i=1,\cdots,k}$ is linearly independent.

The Lagrange Multipliers Theorem in multi-constraint version asserts that if $f(\mathbf{a})$ is local extreme point of f subject to the constraints $g_i = 0$, then there exist $\lambda_1, \cdots, \lambda_k$ such that

$$\nabla f|_{\mathbf{a}} + \sum_{i=1}^{k} \lambda_i \nabla g_i|_{\mathbf{a}} = \mathbf{0}\,. \tag{$\bullet$}$$

Please note that

(i) Suppose the hypotheses hold. If ($\bullet$) is not satisfied then there is no extreme point.

(ii) If ($\bullet$) is not satisfied, then it means nothing. You should check the hypotheses hold in entire domain to determine whether there is extreme point or not.

(iii) If the hypotheses don't hold, then it means nothing.

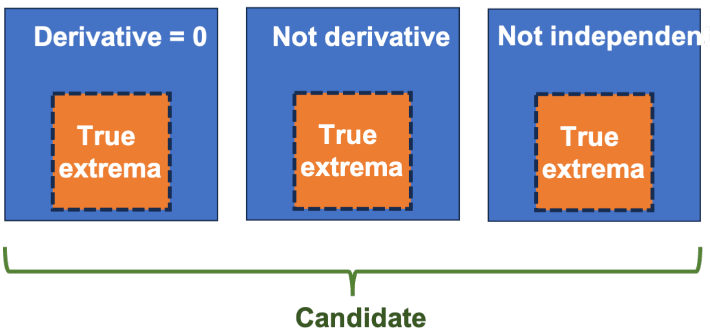
Please see the following exercise.

(a) Let $g(x, y) = x^3 - y$. Minimize or maximize $f(x, y) = y$, which subjects to $g = 0$. Solve ($\bullet$) and $\lambda = 1$ gives extreme value at $(0, 0)$ but there is no minimum or maximum.

(b) Let $g(x, y) = y^2 - x^4 + x^3$. Minimize $f(x, y) = x$, which subjects to $g(x, y) = 0$. The minimum occurs at $(0, 0)$ but $(0, 0)$ is not solution to ($\bullet$). However, $g = 0$ is not differentiable at $(0, 0)$.

(c) Let $g(x, y, z) = x^2 + y^2$. Minimize or maximize $f(x, y, z) = x + y$, which subjects to $g(x, y, z) = 0$, *i.e.* z-axis. Both minimum and maximum occur at $(0, 0, z)$ but $(0, 0, z)$ is not solution to ($\bullet$). However, $\nabla g|_{(0,0,z)} = (0, 0, 0)$ for all z.

(d) Let $g_1(x, y, z) = x^6 - z$, $g_2(x, y, z) = y^3 - z$. Minimize $f(x, y, z) = y$, which subjects to $g_1 = g_2 = 0$. Note that the constraint curve is (t, t^2, t^6). Hence, the minimum occurs at $(0, 0, 0)$ but $(0, 0, 0)$ is not solution to ($\bullet$). However, $\nabla g_1|_{(0,0,0)}$, $\nabla g_2|_{(0,0,0)}$ are not linearly independent.



Please refer to Chapter 5.6 in [P] and Theorem 11.63 in [W] for more details.