# Calculus III TA Session

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1. $\boxed{\text{109-2 (01-05) Midterm Problem 3 (Linear approximation)}}$
   Let $g(x, y, z)$ be a function defined on $\mathbb{R}^3$ with continuous partial derivatives. Suppose that
   $$|\nabla g(2, 1, 3)|^2 = 24 \text{ and } g_z(2, 1, 3) > 0.$$
   Moreover, the trajectories of the two curves
   $$\mathbf{r}_1(s) = \left\langle 2s, s^2, 1 + 2s \right\rangle \text{ and } \mathbf{r}_2(t) = \left\langle 2e^t, \cos t, 3 + t + 5t^2 \right\rangle$$
   lie on the level surface $g(x, y, z) = 0$ completely.
   (a) Find the vector $\nabla g(2, 1, 3)$.
   (b) Suppose that $f(x, y, z)$ is a function defined on $\mathbb{R}^3$ with continuous partial derivatives such that $f(2, 1, 3) \geqslant f(x, y, z)$ for every point $(x, y, z)$ on the level surface $g(x, y, z) = 0$. If $f(2, 1, 3) = 5, |\nabla f(2, 1, 3)|^2 = 6$ and $f_y(2, 1, 3) > 0$, estimate the value of $f(2.01, 0.9, 3.02)$ by the linear approximation of $f$ at $(2, 1, 3)$.

2. $\boxed{\text{109-2 (01-05) Midterm Problem 4 (Second derivative test)}}$
   Let $f(x, y) = \frac{xy(x+y)}{e^{x+y}}$ be defined on the first quadrant $D : x > 0$ and $y > 0$ (without boundary). Find all critical points of $f$ in $D$ and classify them (as local maximum points, local minimum points, or saddle points). Please provide details of calculation.

3. $\boxed{\text{107-2 A Midterm Problem 4 (Second derivative test)}}$
   Find and classify all critical points of $f(x, y) = 4x^3 + 2xy^2 + \frac{2}{3}y^3 + 6x^2$. Reminder: each critical point must be shown to be either a local maximal point, a local minimal point, or neither of the above.

4. $\boxed{\text{111-2 (01-05) Midterm Problem 2 (Lagrange multipliers)}}$
   Let $F(x, y, z) = x^2+y^2+z^2$ and $G(x, y, z) = z^3-3xy+y^2$. Let $C$ be the curve of intersection of the level surfaces $F(x, y, z) = 9$ and $G(x, y, z) = 6$.
   (a) Find a parametization of the tangent line of $C$ at $(1, 2, 2)$.
   (b) Near $(1, 2, 2)$, the curve defines $y = y(x)$ and $z = z(x)$ as differentiable functions in $x$.
      (i) Find $\frac{d}{dx}F(x, y(x), z(x))\big|_{x=1}$ and $\frac{d}{dx}G(x, y(x), z(x))\big|_{x=1}$. Express your answers in $y'(1)$ and $z'(1)$.
      (ii) Hence, find the values of $y'(1)$ and $z'(1)$.
   (c) It is known that a differentiable function $H(x, y, z)$, when restricted to the surface $F(x, y, z) = 9$, attains its absolute maximum value at $(1, 2, 2)$ and $H_y(1, 2, 2) = -2$. Use linearization to estimate the value of $H(1.1, 1.9, 2.1) - H(1, 2, 2)$.

5. $\boxed{\text{111-2 (01-05) Midterm Problem 3 (Lagrange multipliers)}}$
   It 15 known that the plane $x + y - 2z = 5$ and the cylinder $3x^2 + 2xy + 3y^2 = 16$ intersect at an ellipse $\Gamma$ centered at $\left(0, 0, -\frac{5}{2}\right)$. Apply the method of Lagrange multipliers to find the maximum and minimum distances of $\Gamma$ from its center.

6. $\boxed{\text{109-2 (01-05) Midterm Problem 5 (Lagrange multipliers)}}$
A plagen is formed by placing an isosceles triangle on a rectangle. The side lengths are denoted by $a, b$, and $c$ as shown in the figure.

(a) Write down the area of pentagon in terms of $a, b$, and $c$.

(b) Find the maximum area of pentagon if the perimeter is fixed as 2.

7. $\boxed{\text{108-2 A Midterm Problem 3 (Lagrange multipliers)}}$
Let $C$ be the hyperbola formed by the intersection of the cone $x^2+3z^2=4y^2$ and the plane $2x+y=5$. Find the maximum and the minimum distance between the origin and the point on $C$ (if exist) by the method of Lagrange multipliers.

8. $\boxed{\text{107-2 A Midterm Problem 5 (With and without constraint)}}$
Consider the part of an elliptic paraboloid defined by $z=\frac{x^2}{16}+\frac{y^2}{8}, z \leq 6$. Find the points on the surface segment which are respectively the farthest from and the closest to the point $(0,0,8)$.

9. $\boxed{\text{102-2 A1 Midterm Problem 8 (With and without constraint)}}$
Find the maximum and minimum values of $xy+z^2$ on the ball $x^2+y^2+\left(z-\frac{1}{2}\right)^2 \leq 1$.

10. $\boxed{\text{112-2 (11-14) Worksheet Problem 4(c) (Lagrange multipliers)}}$
Define a constraint $P_L L + P_K K = I$, where $P_L, P_K > 0$ are constants. Consider a more general production function $f(L,K)=L^\alpha+K^\alpha$, where $\alpha>0$ is a constant. Does the ratio $\frac{L^*}{K^*}$ remain the same when $I$ varies?

**Solution:** Let $f(L,K)=L^\alpha+K^\alpha$ and $g(L,K)=P_L L+P_K K-I$. We have

$$\begin{cases} \alpha L^{\alpha-1}=\lambda P_L \\ \alpha K^{\alpha-1}=\lambda P_K \\ P_L L+P_K K=I \end{cases}$$

Suppose $L=0$. Since $P_L>0$, $\lambda=0$ and hence $K=0$, which contradicts to $I>0$. Suppose $\lambda \neq 0$. **Case 1:** if $\alpha \neq 1$, then

$$\left(\frac{L}{K}\right)^{\alpha-1}=\frac{P_L}{P_K}$$

and hence

$$\frac{L}{K}=\left(\frac{P_L}{P_K}\right)^{1/(\alpha-1)}$$

independent on $I$. **Case 2:** of $\alpha=1$, we get

$$L^0=\lambda P_L \qquad K^0=\lambda P_K$$

Since $L, K \neq 0$, this case cannot happended. Note that $0^0$ is not well-defined.

**Remark.** If $u$ parallel to $v$ in $\mathbb{R}^2$, there are two cases. First, $u_1=\lambda v_1$ and $u_1=\lambda v_1$; second, $u_1=v_1=0$ or $u_2=v_2=0$.