Calculus Ⅲ TA Session

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1. 109-2 (01-05) Midterm Problem 3 (Linear approximation)

   Let $g(x, y, z)$ be a function defined on $\mathbb{R}^3$ with continuous partial derivatives. Suppose that

   $$|\nabla g(2,1,3)|^2 = 24 \text{ and } g_z(2,1,3) > 0.$$

   Moreover, the trajectories of the two curves

   $$\mathbf{r}_1(s) = \langle 2s, s^2, 1+2s \rangle \text{ and } \mathbf{r}_2(t) = \langle 2e^t, \cos t, 3+t+5t^2 \rangle$$

   lie on the level surface $g(x, y, z) = 0$ completely.

   (a) Find the vector $\nabla g(2, 1, 3)$.

   (b) Suppose that $f(x, y, z)$ is a function defined on $\mathbb{R}^3$ with continuous partial derivatives such that $f(2, 1, 3) \geqslant f(x, y, z)$ for every point (x, y, z) on the level surface $g(x, y, z) = 0$. If $f(2, 1, 3) = 5, |\nabla f(2, 1, 3)|^2 = 6$ and $f_y(2, 1, 3) > 0$, estimate the value of $f(2.01, 0.9, 3.02)$ by the linear approximation of f at $(2, 1, 3)$.

2. 107-2 A Midterm Problem 4 (Second derivative test)

   Find and classify all critical points of $f(x, y) = 4x^3 + 2xy^2 + \frac{2}{3}y^3 + 6x^2$. Reminder: each critical point must be shown to be either a local maximal point, a local minimal point, or neither of the above.

3. 109-2 (01-05) Midterm Problem 5 (Lagrange multipliers)

   A plagen is formed by placing an isosceles triangle on a rectangle. The side lengths are denoted by a, b, and c as shown in the figure.

   (a) Write down the area of pentagon in terms of a, b, and c.

   (b) Find the maximum area of pentagon if the perimeter is fixed as 2.

4. 107-2 A Midterm Problem 5 (Mixed)

   Consider the part of an elliptic paraboloid defined by $z = \frac{x^2}{16} + \frac{y^2}{8}, z \leq 6$. Find the points on the surface segment which are respectively the farthest from and the closest to the point $(0, 0, 8)$.

5. 102-2 A1 Midterm Problem 8 (With and without constraint)

   Find the maximum and minimum values of $xy + z^2$ on the ball $x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 \leq 1$.

6. 109-2 (13-17) Midterm Problem 4 (Double integral)

   Sketch the region of integration, change the order of integration, and evaluate it.

   (a) $\int_0^4 \int_{\sqrt{x}}^2 \sin\left(\frac{x}{y}\right) \mathrm{d}y \, \mathrm{d}x$.

   (b) $\int_1^2 \int_{\sqrt{y}}^y \cos\left(\frac{x^3}{3} - \frac{x^2}{2}\right) \mathrm{d}x \, \mathrm{d}y + \int_2^4 \int_{\sqrt{y}}^2 \cos\left(\frac{x^3}{3} - \frac{x^2}{2}\right) \mathrm{d}x \, \mathrm{d}y$.

7. $\boxed{\text{112-2 (11-14) Worksheet Problem 4(c) (Lagrange multipliers)}}$

Define a constraint $P_L L + P_K K = I$, where $P_L, P_K > 0$ are constants. Consider a more general production function $f(L, K) = L^\alpha + K^\alpha$, where $\alpha > 0$ is a constant. Does the ratio $\frac{L^*}{K^*}$ remain the same when I varies?

Solution: Let $f(L, K) = L^\alpha + K^\alpha$ and $g(L, K) = P_L L + P_K K - I$. We have

$$\begin{cases} \alpha L^{\alpha-1} = \lambda P_L \\ \alpha K^{\alpha-1} = \lambda P_K \\ P_L L + P_K K = I \end{cases}$$

Suppose $L = 0$. Since $P_L > 0$, $\lambda = 0$ and hence $K = 0$, which contradicts to $I > 0$. Suppose $\lambda \neq 0$. **Case 1:** if $\alpha \neq 1$, then

$$\left(\frac{L}{K}\right)^{\alpha-1} = \frac{P_L}{P_K}$$

and hence

$$\frac{L}{K} = \left(\frac{P_L}{P_K}\right)^{1/(\alpha-1)}$$

independent on I. **Case 2:** of $\alpha = 1$, we get

$$L^0 = \lambda P_L \qquad K^0 = \lambda P_K$$

From our argument, we know that $L, K = 0$ cannot be happended. Note that 0^0 is not well-defined. Hence, since $L, K \neq 0$,

$$\frac{1}{\lambda} = P_L = P_K\,.$$

Then,

$$f(L, K) = L + K = \frac{I}{P_L}$$

and hence L/K may change when I changes.

Remark. If u parallel to v in $\mathbb{R}^2$, there are two cases. First, $u_1 = \lambda v_1$ and $u_2 = \lambda v_2$; second, $u_1 = v_1 = 0$ or $u_2 = v_2 = 0$.