Calculus Ⅲ TA Session

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1. **(Differentiable)** 1102 (01-05) Midterm Problem 1

   Let $f(x,y) = \begin{cases} \frac{x^2y^2}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

   (a) Is $f(x,y)$ continuous at $(0,0)$? Explain.

   (b) Find $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$.

   (c) Write down the linearization $L(x,y)$ of $f(x,y)$ at $(0,0)$.

   (d) The function f is differentiable at $(0,0)$ if

   $$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - L(x,y)}{\sqrt{x^2+y^2}} = 0$$

   where $L(x,y)$ is the linearization of $f(x,y)$ at $(0,0)$. Is $f(x,y)$ differentiable at $(0,0)$? Explain.

   (e) Find $f_y(x,y)$ when $(x,y) \neq (0,0)$. Is $f_y(x,y)$ continuous at $(0,0)$? Explain.

2. **(Chain Rule)** 1112 (11-14) Midterm Problem 2

   The graph $z = f(x,y)$ of the differentiable function f has $2x-3y+z=4$ as its tangent plane at the point $(0,0,4)$. The graph $z = g(x,y)$ of the differentiable function g has $x+2y-z=3$ as its tangent plane at the point $(0,0,-3)$. Answer the following questions.

   (a) Determine the values: $f(0,0), f_x(0,0), f_y(0,0), g(0,0), g_x(0,0), g_y(0,0)$.

   (b) Use the linearization of f at $(0,0)$ to estimate $f(0.1,-0.1)$.

   (c) Let $h(u,v) = ue^{-2v}$ and $u = f(x,y), v = g(x,y)$. Use the Chain Rule to find the partial derivative

   $$\frac{\partial}{\partial x} h(f(x,y), g(x,y)) \text{ at } x = 0, y = 0$$

3. **(linearization)** 111-2 (01-05) Midterm Problem 2

   Let $F(x,y,z) = x^2+y^2+z^2$ and $G(x,y,z) = z^3-3xy+y^2$. Let C be the curve of intersection of the level surfaces $F(x,y,z) = 9$ and $G(x,y,z) = 6$.

   (a) Find a parametization of the tangent line of C at $(1,2,2)$.

   (b) Near $(1,2,2)$, the curve defines $y = y(x)$ and $z = z(x)$ as differentiable functions in x.

   (i) Find $\frac{d}{dx}F(x,y(x),z(x))\big|_{x=1}$ and $\frac{d}{dx}G(x,y(x),z(x))\big|_{x=1}$. Express your answers in $y'(1)$ and $z'(1)$.

   (ii) Hence, find the values of $y'(1)$ and $z'(1)$.

   (c) It is known that a differentiable function $H(x,y,z)$, when restricted to the surface $F(x,y,z) = 9$, attains its absolute maximum value at $(1,2,2)$ and $H_y(1,2,2) = -2$. Use linearization to estimate the value of $H(1.1,1.9,2.1) - H(1,2,2)$.

4. **(Extreme values)** 1112 (11-14) Midterm Problem 4

   Let $f(x,y) = (2-y)\left(x^2+4y^2\right)$.

   (a) Find all critical points of $f(x,y)$ and determine which is a saddle point or gives a local maximum or local minimum.

(b) Let $D = \{(x,y) \mid x^2 + y^2 \le 4\}$. Find the absolute maximum and minimum values of $f(x,y)$ on D.

5. **(Double integral)** 109-2 (13-17) Midterm Problem 4
Sketch the region of integration, change the order of integration, and evaluate it.

   (a) $\int_0^4 \int_{\sqrt{x}}^2 \sin\left(\frac{x}{y}\right) \, dy \, dx$.

   (b) $\int_1^2 \int_{\sqrt{y}}^{y} \cos\left(\frac{x^3}{3} - \frac{x^2}{2}\right) dx \, dy + \int_2^4 \int_{\sqrt{y}}^{2} \cos\left(\frac{x^3}{3} - \frac{x^2}{2}\right) dx \, dy$.

6. Let $f(x,y) = e^{x^2 y}$.

   (a) Find its gradient at the point $(2,0)$.

   (b) At the point $(2,0)$, find the directional derivative along the direction $\frac{1}{\sqrt{5}}(2,-1)$.

   (c) At the point $(2,0)$, find the tangent line of its level curve.

   (d) Approximate the value $f(2.02, -0.01)$.

7. Let $X_i \sim exp(\lambda)$. That is, $f_X(x) = \lambda e^{-\lambda x}$. Find the p.d.f. $Z = X_1 + X_2$. Consider the gamma function $X \sim Gamma(\alpha, \lambda)$ satisfied $f(x \mid \alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot x^{\alpha - 1} \cdot e^{-x\lambda}$.