A DIRTY NOTE OF COVARIANT DERIVATIVE (VERSION 2)

Remark: Section order is rearranged in this note. Here, this note tells a story to help you fill the gap of geometry between 2-dimension surface and higher dimension. There are many concrete examples in this note, which reduce the higher dimensional geometric concept to 2-dimensional surface.

1. Starting from first fundamental form

In November 1915, Einstein presented the theory of general relativity. This essay show the relation between shape of our spacetime and energy-momentum $T_{\mu\nu}$ in our spacetime must satisfies a equation, which is known as Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \,, \tag{1}$$

where $R_{\mu\nu}$ and R are Ricci curvature and scalar curvature, respectively. Note that the shape of spacetime is determined by first fundamental form. On the other hand, $R_{\mu\nu}$ and R are function of first fundamental form $g_{\mu\nu}$. That is, given $T_{\mu\nu}$, the equation $F(g_{\alpha\beta}) = T_{\mu\nu}$ could solved for first fundamental form $g_{\mu\nu}$ for our spacetime.

In March 1916, Schwartzschild found the first solution to Einstein equation (1) which is restricted to spherically symmetric spacetime. The well-known solution is called Schwartzschild metric

$$g_{\mu\nu} = -(1 - \frac{2GM}{c^2 r}) c^2 dt^2 + \frac{1}{1 - \frac{2GM}{c^2 r}} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\theta^2 \,.$$

Note that Riemann geometry is presented in 1845. This is a non-Euclidean geometry, which rejects the validity of Euclid's fifth postulate[1]. However, Riemann geometry was neglected until general relativity was born.

2. Why intrinsic?

Hence, the first fundamental form might be **usually**[2] found first, which might be different from most people thoughts: parametrization first. Then, many geometric quantity have to be developed to measure the shape of manifold. These geometric quantity is only derived from the first fundamental form. In other words, it is no need to put the manifold in higher dimensional

[1]see section 7 the last page.

[2]The connection can be chosen without metric, see section 3.

Euclidean space to know what shape it is[3]. For instance, the Gauss-Bonnet Theorem says given a geodesic triangle ($\kappa_g = 0$),

$$\sum_{i=1}^{3} \psi_i - \pi = \iint K d\sigma \,,$$

where ψ_i is inner angle and K is Gaussian curvature. Moreover, according to Gaussian Egregium theorem, Gaussian curvature is only depend on first fundamental form. Thus, given the first fundamental form, we could **imagine** the manifold looks like **without parametrization**.

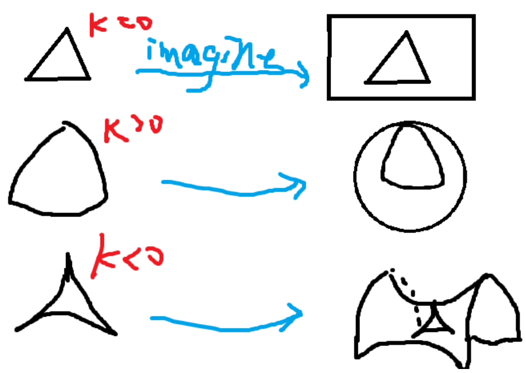

FIGURE 1

Thus, Gaussian curvature is a intrinsic geometric quantity. It can be derived from the first fundamental form and it build the relation between the first fundamental form and the shape of manifold. For instance, could you know the shape of following form:

(1) $dx^2 + dy^2$

(2) $dr^2 + r^2 d\theta^2$

(3) $dz^2 + d\theta^2$

(4) $a^2 \cosh v du^2 + a^2 \cosh^2 v dv^2$.

Question: given an arbitrary metric, does it exist only one parametrization? how can we define derivative on it?

3. Covariant derivative

Recall that in your textbook, the covariant derivative is defined as

$$\nabla_u w = \left(\frac{\partial w}{\partial u}\right)^T ,$$

[3]By Whitney's embedding theorem, every manifold can be embedded in higher dimensional Euclidean space.

where w is a vector on surface *i.e.* $w = a\sigma_u + b\sigma_v$. Now, given a metric, how to build the derivative on manifold **without higher dimensional Euclidean space**. Another view of covariant derivative is introduced as following. The main idea is derivative with respect to the **moving basis**. For instance, given $g = dr^2 + r^2 d\theta^2$, let vector $\sigma_\theta(1, \frac{\pi}{4})$. What is $\nabla_\theta \sigma_\theta(1, \frac{\pi}{4})$? In fact,

$$\begin{aligned}
\nabla_\theta \sigma_\theta(1, \frac{\pi}{4}) &= (\sigma_{\theta\theta}(1, \frac{\pi}{4}))^T && \text{This is in your textbook.} \\
&= (\Gamma^r_{\theta\theta}\sigma_r + \Gamma^\theta_{\theta\theta}\sigma_\theta + N\mathbf{N})^T \\
&= \Gamma^r_{\theta\theta}\sigma_r + \Gamma^\theta_{\theta\theta}\sigma_\theta && \text{I will say this is differentiate basis.} \\
&= \frac{-1}{r}\sigma_r
\end{aligned}$$

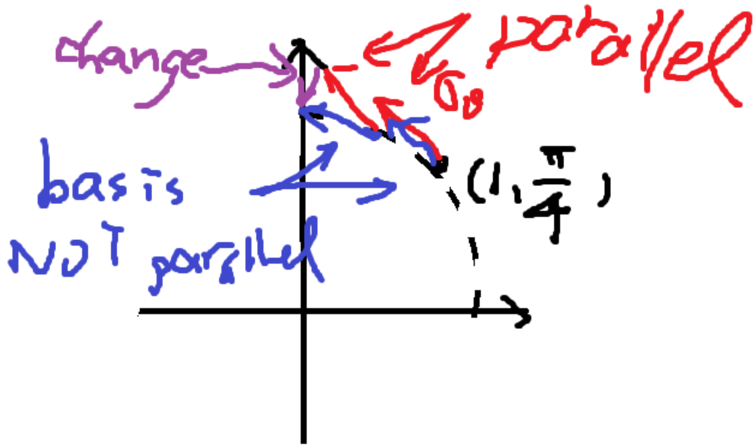

FIGURE 2

Thus, covariant derivative is contained two part: **one is derivative the scalar and the other is derivative the basis**. For convention, write $\nabla_u = \partial_u + \Gamma_u$ where ∂_u does not differentiate basis. That is, given a vector $w = w^i\sigma_i$, $\nabla_u(w^i\sigma_i) = (\partial_u w^i)\sigma_i + w^i\Gamma^\lambda_{ui}\sigma_\lambda = (\partial_u w^\lambda)\sigma_\lambda + w^i\Gamma^\lambda_{ui}\sigma_\lambda$. Hence, **covariant** can explain that it measure the change between two vector. Note that when an index variable appears twice in a single term, it implies summation of that term over all the values of the index, *i.e.* $w^i\sigma_i = \sum_i w^i\sigma_i$ Above convention is known as **Einstein convention**.

On the other hand, **Gauss formula** gives a formula of Christoffel symbol depend on the first fundamental form.

$$
\begin{aligned}
\sigma_{uu} &= \Gamma^u_{uu}\sigma_u + \Gamma^v_{uu}\sigma_v + L\mathbf{N} \\
\sigma_{uv} &= \Gamma^u_{uv}\sigma u + \Gamma^v_{uv}\sigma_v + M\mathbf{N} \\
\sigma_{vv} &= \Gamma^u_{vv}\sigma_u + \Gamma^v_{vv}\sigma_v + N\mathbf{N}
\end{aligned}
$$

where

$$
\begin{aligned}
\Gamma^u_{uu} &= \frac{GE_u - 2FF_u + FE_v}{2\left(EG - F^2\right)} \\
\Gamma^v_{uu} &= \qquad \cdots \\
&\vdots
\end{aligned}
$$

Check the textbook, you can find the remainder terms. The proof is used $\langle \sigma_{uu}, \sigma_u \rangle$, $\langle \sigma_{uu}, \sigma_v \rangle$ and $\langle \sigma_{uu}, \mathbf{N} \rangle$ and so on. Remark that the differentiation on manifold can be calculated **without parametrization**.

The formula of Christoffel symbol might be remember[4]. Hence, the general formula is given as follows and hope it can help you to remember the formula,

$$
\Gamma^k_{ij} = \frac{1}{2} g^{k\xi}(\partial_i g_{j\xi} + \partial_j g_{i\xi} - \partial_\xi g_{ij})
$$

where g_{ij} is component of metric and g^{ij} is component of inverse matrix of metric

$$
g^{ij} = (g_{ab})^{-1} = \frac{1}{EG - F^2} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix}.
$$

Note that I usually use ξ to be a dummy variable, *i.e.* $g^{i\xi}a_\xi = \sum_\xi g^{i\xi} v_\xi$.

Example 3.1. *Given metric $g = Edu^2 + 2Fdudv + Gdv^2$, find the Christoffel symbol Γ^u_{uu},*

$$
\begin{aligned}
\Gamma^u_{uu} =& \frac{1}{2} g^{uu}(\partial_u g_{uu} + \partial_u g_{uu} - \partial_u g_{uu}) && \xi = u \\
&+ \frac{1}{2} g^{uv}(\partial_u g_{uv} + \partial_u g_{uv} - \partial_v g_{uu}) && \xi = v \\
=& \frac{G}{2(EG - F^2)}(E_u + E_u - E_u) + \frac{-F}{2(EG - F^2)} F(F_u + F_u - E_v) \\
=& \frac{GE_u - 2FF_u + FE_v}{2\left(EG - F^2\right)}.
\end{aligned}
$$

[4] In graduate school

Moreover, this Christoffel symbol is called Levi-Civita connection. In fact, there is many choice of connection. However, there are more geometric meaning when using Levi-Civita connection. This connection is satisfies two properties:

(1) competible with metric *i.e.* $\nabla_i g(u,v) = g(\nabla_i u, v) + g(u, \nabla_i v)$

(2) torsion free.

4. Constraint of surface in $\mathbb{R}^3$

We know the 2 dimension manifold in $\mathbb{R}^3$ must satisfies

$$\begin{cases} \sigma_{ij} = \Gamma^k_{ij}\sigma_k + A_{ij}\mathbf{N} \\ \mathbf{N}_i = -a_{1i}\sigma_u - a_{2i}\sigma_v = -g^{jk}A_{ij}\sigma_k \end{cases} .$$

Question: Given two matrix P and Q, is it exist $\sigma : (u,v) \mapsto S \subset \mathbb{R}^3$ such that $F_I = g = P$ and $F_{II} = A = Q$. Does it has any constraint of the first and second fundamental form? Yes! there is a constraint, which is known as **Gauss-Codazzi equation**. By $(\sigma_{ij})_k = (\sigma_{ik})_j$, then

$$\text{(Gauss equation)} \begin{cases} EK = (\Gamma^v_{uu})_v - (\Gamma^v_{uv})_u + \Gamma^u_{uu}\Gamma^v_{uv} + \Gamma^v_{uu}\Gamma^v_{vv} - \Gamma^u_{uv}\Gamma^v_{uu} - (\Gamma^v_{uv})^2 \\ FK = \cdots \\ GK = \cdots \end{cases}$$

and

$$\text{(Codazzi equation)} \begin{cases} L_v - M_u = L\Gamma^u_{uv} + M\left(\Gamma^v_{uv} - \Gamma^u_{uu}\right) - N\Gamma^v_{uu} \\ M_v - N_u = \cdots \end{cases}$$

According to Gauss-Codazzi equation, the **Gauss Egregium theorem** says Gaussian curvature only depend on the first fundamental form *i.e.* $K = F(g_{ij}, \partial g_{ij})$. That is, **Gaussian curvature is invariant under isometry**.

5. Riemann curvature

In order to know the geometric meaning of Gauss-Codazzi equation, the Riemann curvature have to be introduced. As your HW7, define[5] Riemann curvature tensor as

$$R(x,y)w = \nabla_x\nabla_y w - \nabla_y\nabla_x w\,,$$

[5] Note that the definition in do Carmo's textbook is different from John Lee's.

where x, y, w are vector. Above definition is **NOT** correct because it is lack of $-\nabla_{[\sigma_u,\sigma_v]}\sigma_u$. However, the coordinate is usually chosen for $[\partial_u, \partial_v] = 0$. Hence, this term is dropped here. The definition can be rewrite in local coordinate form,

$$R(\sigma_i, \sigma_j)\sigma_k = R_{ijk}{}^{l}\sigma_l \,,$$

and the Riemann curvature tensor can be also defined as

$$R_{ijkl} = g_{l\xi}R_{ijk}{}^{\xi} = \langle \nabla_i\nabla_j\sigma_k - \nabla_j\nabla_i\sigma_k, \sigma_l\rangle$$

Now, the following explain is very **dirty**. *Don't ask any question about the following explain.* Given a vector w,

$$\begin{aligned}
R(\sigma_u, \sigma_v)w = \lim_{r,s\to 0} \frac{w - DCBAw}{rs} &= \lim_{r,s\to 0} DC\frac{1}{rs}\left(C^{-1}D^{-1}w - BAw\right) \\
&= \frac{1}{rs}\left(C^{-1}\left(D^{-1}w - w\right) + \left(C^{-1}\vec{w} - \vec{w}\right)\right) - \frac{1}{rs}(B(Aw - w) + (Bw - w)) \\
&= \lim_{r,s\to 0} DC\left[\frac{C^{-1}}{r}(\nabla_v w) + \frac{1}{s}\nabla_u w - \frac{B}{s}(\nabla_u w) - \frac{1}{r}\nabla_v w\right] \\
&= \lim_{r,s\to 0} DC\left[\frac{C^{-1}\nabla w - \nabla_v w}{r}\right] - \left[\frac{B\nabla_u w - \nabla_u w}{s}\right] \\
&= DC(\nabla_u\nabla_v w - \nabla_v\nabla_u w)
\end{aligned}$$

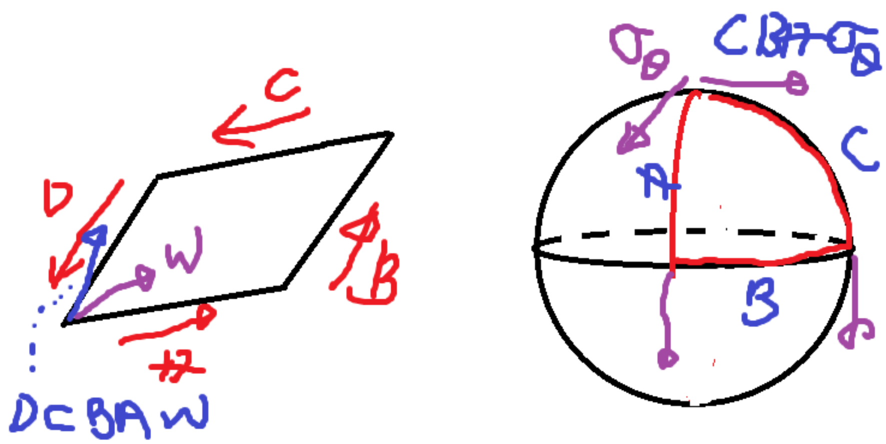

FIGURE 3

Example 5.1. *Let* $\{\sigma_\theta, \sigma_v\}$ *be basis of* $\mathbb{S}^2$ *and the metric is* $g = d\theta^2 + \sin^2\theta\, d\phi^2$*. The Christoffel symbol is as following*

$$
\begin{array}{lllll}
\Gamma_\theta\text{:} & \Gamma^\theta_{\theta\theta}=0 & \Gamma^\phi_{\theta\theta}=0 & \Gamma^\theta_{\theta\phi}=0 & \Gamma^\phi_{\theta\phi}=\cot\theta \\
\Gamma_\phi\text{:} & \Gamma^\theta_{\phi\theta}=0 & \Gamma^\phi_{\phi\theta}=\cot\theta & \Gamma^\theta_{\phi\phi}=-\sin\theta\cos\theta & \Gamma^\phi_{\phi\phi}=0
\end{array}
$$

Hence,

$$
\begin{aligned}
\nabla_\theta\nabla_\phi\sigma_\theta &= \partial_\theta\nabla_\phi\sigma_\theta - \Gamma^\xi_{\theta\phi}\nabla_\xi\sigma_\phi + \Gamma^\xi_{\theta\theta}\nabla_\phi\sigma_\xi \\
&= \partial_\theta\Gamma^\lambda_{\phi\theta}\sigma_\lambda - \Gamma^\xi_{\theta\phi}\Gamma^\lambda_{\xi\phi}\sigma_\lambda + \Gamma^\xi_{\theta\theta}\Gamma^\lambda_{\phi\xi}\sigma_\lambda \\
&= \partial_\theta\Gamma^\phi_{\phi\theta}\sigma_\phi - \Gamma^\phi_{\theta\phi}\Gamma^\theta_{\phi\phi}\sigma_\theta
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\nabla_\phi\nabla_\theta\sigma_\theta &= \partial_\phi\Gamma^\lambda_{\theta\theta}\sigma_\lambda - \Gamma^\xi_{\phi\theta}\Gamma^\lambda_{\xi\phi}\sigma_\lambda + \Gamma^\xi_{\phi\theta}\Gamma^\lambda_{\theta\xi}\sigma_\lambda \\
&= -\Gamma^\phi_{\phi\theta}\Gamma^\theta_{\phi\phi}\sigma_\theta + \Gamma^\phi_{\phi\theta}\Gamma^\phi_{\theta\phi}\sigma_\phi
\end{aligned}
$$

Hence,

$$
R_{\phi\theta\theta\phi} = \langle \nabla_\phi\nabla_\theta\sigma_\theta - \nabla_\theta\nabla_\phi\sigma_\theta, \sigma_\phi\rangle = \langle \Gamma^\phi_{\phi\theta}\Gamma^\phi_{\theta\phi}\sigma_\phi - \partial_\theta\Gamma^\phi_{\phi\theta}\sigma_\phi, \sigma_\phi\rangle = \sin^2\theta
$$

Note that write $\nabla_u = \partial_u + \Gamma_u$ where ∂_u does not differentiate basis, for convention. Remark that the first minus is because ∇ is a differential form. Corresponding to HW7,

$$
K = \frac{\langle \nabla_\phi\nabla_\theta\sigma_\theta - \nabla_\theta\nabla_\phi\sigma_\theta, \sigma_\phi\rangle}{EG-F^2} = \frac{\sin^2\theta}{\sin^2\theta} = 1
$$

In fact, according to above calculation, the formula of Riemann curvature is follows

$$
R_{ijk}{}^{l} = \partial_i\Gamma^l_{jk} - \partial_j\Gamma^l_{ik} - \Gamma^\xi_{ik}\Gamma^l_{j\xi} + \Gamma^\xi_{jk}\Gamma^l_{i\xi}\,,
$$

and

$$
R_{ijkl} = g_{l\lambda}R_{ijk}{}^{\lambda}
$$

I remember above formula as $R_{ijk}{}^{l} = \partial_i\Gamma_j - \partial_j\Gamma_i - \Gamma_i\Gamma_j + \Gamma_j\Gamma_i$ and filled in k, l by the same position.

Now, it is sufficient to know the geometric meaning of Gauss-Codazzi equation:

$$
R_{kji}{}^{l} = g^{l\xi}\left(A_{ij}A_{k\xi} - A_{ik}A_{j\xi}\right)
$$

and

$$
\partial_k A_{ij} - \partial_j A_{ik} + \Gamma^\xi_{ij}A_{\xi k} - \Gamma^\xi_{ik}A_{\xi j} = 0\,.
$$

6. Parallel transport

Given a vector $w(u,v) = a(u,v)\sigma_u + b(u,v)\sigma_v$ and a path $\gamma(u,v) = \sigma(u(t),v(t))$ with $\gamma' = u'\sigma_u + v'\sigma_v$. Then

$$\begin{aligned}\frac{dw}{dt} &= \frac{d}{dt}(a\sigma_u + b\sigma_v) \\ &= a'u'\sigma_u + a[(\Gamma^u_{uu}\sigma_u + \Gamma^v_{uu}\sigma_v + L\mathbf{N})u' + (\Gamma^u_{uv}\sigma_u + \Gamma^v_{uv}\sigma_v + M\mathbf{N})v'] + b'\sigma_v + b[\cdots]\end{aligned}$$

Hence, the following can be defined

$$\frac{Dw}{dt} = \left(\frac{dw}{dt}\right)^T = a'u'\sigma_u + a[(\Gamma^u_{uu}\sigma_u + \Gamma^v_{uu}\sigma_v)u' + (\Gamma^u_{uv}\sigma_u + \Gamma^v_{uv}\sigma_v)v'] + b'\sigma_v + b[\cdots] \tag{2}$$

On the other hand,

$$\begin{aligned}\nabla_{\gamma'}w &= \nabla_{(u'\sigma_u + v'\sigma_v)}(a\sigma_u + b\sigma_v) \\ &= u'\nabla_u(a\sigma_u + b\sigma_v) + v'\nabla_v(a\sigma_u + b\sigma_v) \\ &= u'a'\sigma_u + u'a\Gamma^\xi_{uu}\sigma_\xi + \cdots \qquad \text{This is equal to (2)} \\ &= \left(\frac{dw^\xi}{dt} + \frac{d\gamma^i}{dt}w^i\Gamma^\xi_{ij}\right)\sigma_\xi = 0\end{aligned}$$

where $\gamma^u = u$, $\gamma^v = v$, $w^u = a$ and $w^v = b$. The example is given in textbook Example 7.4.7 pp 147-148.

7. Geodesic

Let path $\gamma(u,v) = \sigma(u(t),v(t))$ with $\gamma' = u'\sigma_u + v'\sigma_v$. By above computation, then

$$\frac{D\gamma'}{ds} = \nabla_{\gamma'}\gamma' = \left(\frac{d^2x^\xi}{ds^2} + \Gamma^\xi_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\right)\sigma_\xi = 0$$

Finally, we compute two example.

Example 7.1. *Let metric $g = d\theta^2 + \sin^2\theta d\phi^2$. Let $\xi = theta$*

$$\ddot\theta + \Gamma^\theta_{\phi\phi}\dot\phi^2 = \ddot\theta - \sin\theta\cos\theta\dot\phi^2 = 0 \tag{3}$$

and $\xi = \phi$

$$\ddot\phi + \Gamma^\phi_{\phi\theta}\dot\phi\dot\theta + \Gamma^\phi_{\theta\phi}\dot\theta\dot\phi = \ddot\phi + 2\cot\theta\dot\theta\dot\phi = 0 \tag{4}$$

Hence, the following equations are geodesic equation.

$$\begin{cases}\ddot\theta - \sin\theta\cos\theta\,\dot\phi^2 = 0 \\ \ddot\phi + 2\cot\theta\,\dot\theta\dot\phi = 0\end{cases}$$

Now, the follows is brief explanation why above equation is great circle. Let the curve be constant speed, $\dot\theta^2 + \sin^2\theta\dot\phi^2 = 1$. By (4), $\frac{d}{ds}(\sin^2\theta\dot\phi) = 0$. Let $\sin^2\theta\dot\phi = J = \sin\theta_0$. Then, $\dot\theta^2 = 1 - \frac{J^2}{\sin^2\theta}$. Hence,

$$\frac{d\theta}{ds} = \sqrt{\frac{\sin^2\theta - \sin^2\theta_0}{\sin^2\theta}}$$

implies $s = \sin^{-1}\frac{\cos\theta}{\cos\theta_0}$,

$$\cos\theta = \sin s\cos\theta_0\,.$$

On the other hand, $\sin^2\theta = 1 - \sin^2 s\cos^2\theta_0$

$$\frac{d\phi}{ds} = \frac{\sin\theta_0}{1 - \sin^2 s\cos^2\theta_0}$$

implies $\phi = \tan^{-1}\sin\theta_0\tan s$,

$$\tan\phi = \sin\theta_0\tan s$$

Therefore, the following equation is great circle

$$\begin{cases} \cos\theta = \sin s\cos\theta_0 \\ \tan\phi = \sin\theta_0\tan s \end{cases}$$

Example 7.2. *The second example is in your HW7, Find a revolution surface with $K = -1$.*

$$\begin{cases} f'' - f = 0 \\ f'^2 + g'^2 = 1 \end{cases}$$

Then, $f = ae^v + be^{-v}$. In particular, take $a = 1$, $b = 0$.

$$\begin{aligned} g(v) &= \int\sqrt{1 - e^{2v}}dv = -\int\frac{\sin^2\theta}{\cos\theta}d\theta = \sin\theta - \log(\sec\theta + \tan\theta) \\ &= \sqrt{1 - e^{2v}} - \log(e^{-v} + \sqrt{e^{-2v} - 1}) \\ &= \sqrt{1 - e^{2v}} - \cosh^{-1}(e^{-v})) \end{aligned}$$

Hence, $\sigma(u,v) = (e^v\cos u, e^v\sin u, \sqrt{1 - e^{2v}} - \cosh^{-1}(e^{-v}))$. Let $w = e^{-v}$, $\sigma(u,v) = (\frac{1}{w}\cos u, \frac{1}{w}\sin u, \sqrt{1 - \frac{1}{w^2}} - \cosh^{-1}(w))$. Hence, the first fundamental form is

$$\frac{du^2 + dw^2}{w^2}$$

It's sufficient to compute the geodesic equation

$$\begin{aligned} &\Gamma_u\text{:} \quad \Gamma^u_{uu} = 0 \quad \Gamma^w_{uu} = \tfrac{1}{w} \quad \Gamma^u_{uw} = \tfrac{-1}{w} \quad \Gamma^w_{uw} = 0 \\ &\Gamma_w\text{:} \quad \Gamma^u_{wu} = \tfrac{-1}{w} \quad \Gamma^w_{wu} = 0 \quad \Gamma^u_{ww} = 0 \quad \Gamma^w_{ww} = \tfrac{-1}{w} \end{aligned}$$

Hence,

$$\begin{cases} \ddot{u} - \frac{2}{w}\dot{u}\dot{w} = 0 \\ \ddot{w} + \frac{1}{w}\dot{u}^2 - \frac{1}{w}\dot{w}^2 = 0 \end{cases}$$

Now, try to solve the ode system. If $\dot{u} = 0$ then u is constant. If $\dot{u} \neq 0$ then $\ddot{u} - \frac{2}{w}\dot{u}\dot{w} = \frac{d}{ds}\log\left(\frac{\dot{u}}{w^2}\right) = 0$ implies $\dot{u} = cw^2$. Let $\dot{u}^2 + \dot{w}^2 = w^2$ then $(u-a)^2 + w^2 = \frac{1}{c^2}$. **Question:** Given a straight line (geodesic), could we draw at least two straight lines through a point. The point is not on a given straight line? Do you know the Euclid's fifth postulate?

References

[1] Andrew Pressley, *Elementary Differential Geometry.*

[2] Mantredo do Carmo, *Differential Geomtry of Curves and Surface.*

[3] Mantredo do Carmo, *Riemannian Geometry.*

[4] John Lee, *Introduction to Smooth Manifolds.*

[5] John Lee, *Riemannian Manifolds: An Introduction to Curvature.*

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[9] Maopei Tsui, *Lecture Note in Introduction to Geometry.*

[10] Peiming Ho, *Lecture Note in General Relativity.*