INTRODUCTION TO DIFFERENTIAL GEOMETRY FINAL EXAM SOLUTION (VERSION 2)

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(1) Let $\sigma : U \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ be a smooth local patch, write $\sigma(x_1, \cdots, x_n) = (\sigma_1(x), \cdots \sigma_n(x))$. we define tangent space $T_pS = span\{\frac{\partial \sigma}{\partial x_i}\}_{i=1}^n$ and $N(x)$ is smooth normal vector field. Recall metric tensor is $g_{ij} = \left\langle \frac{\partial \sigma}{\partial x_i}, \frac{\partial \sigma}{\partial x_j} \right\rangle$ and the second fundamental form is $h_{ij} = \left\langle \frac{\partial^2 \sigma}{\partial x_i \partial x_j}, N \right\rangle$. Futhermore, the Weingarten map $\mathcal{W}$ is linear map define on T_pS by

$$\mathcal{W}\left(a^i \frac{\partial \sigma}{\partial x_i}\right) = a^i \mathcal{W}\left(\frac{\partial \sigma}{\partial x_i}\right) = -a^i \frac{\partial N}{\partial x_i}.$$

Let $[\mathcal{W}]_{\mathfrak{B}}$ be matrix representation of Weingarten map with respect to the basis of tangent space $\mathfrak{B} = \{\frac{\partial \sigma}{\partial x_i}\}_i$, *i.e.* $\mathcal{W}\left(\frac{\partial \sigma}{\partial x_i}\right) = W^i_j \frac{\partial \sigma}{\partial x_i}$ with $[\mathcal{W}]_{\mathfrak{B}} = [W^i_j]$

(a) 10 pt Prove that $[\mathcal{W}]_{\mathfrak{B}} = \mathcal{F}_{\mathrm{I}}^{-1}\mathcal{F}_{\mathrm{II}}$, where $(\mathcal{F}_{\mathrm{I}})_{ij} = g_{ij}$ is $n \times n$ matrix and $(\mathcal{F}_{\mathrm{II}})_{ij} = h_{ij}$ is $n \times n$ matrix as well.

(b) 10 pt Show that

$$\left\langle \frac{\partial^2 \sigma}{\partial x^i \partial x^j}, \frac{\partial \sigma}{\partial x^k} \right\rangle = \frac{1}{2}\left(\frac{\partial}{\partial x^i} g_{jk} + \frac{\partial}{\partial x^j} g_{ki} - \frac{\partial}{\partial x^k} g_{ij}\right).$$

(c) 10 pt Recall the Christoffel symbols $\Gamma_{ij}^{\ k}$ is defined by $\frac{\partial^2 \sigma}{\partial x_i \partial x_j} = \Gamma_{ij}^{\ k} \frac{\partial \sigma}{\partial x_k} + h_{ij}N$. Prove that

$$\Gamma_{ij}^{\ k} = \frac{1}{2} g^{kl}\left(\frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{li}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l}\right)$$

where g^{ij} is inverse of g_{ij}, *i.e.* $g^{ik}g_{kj} = \delta^i_j$.

(d) 10 pt Recall the covariant derivative $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \Gamma_{ij}^{\ p} \frac{\partial}{\partial_p}$ where denote $\frac{\partial \sigma}{\partial x_i}$ as $\frac{\partial}{\partial x_i}$. Define the curvature tensor

$$R_{ijk}^{\ \ \ l} \frac{\partial}{\partial x_l} = \nabla_{\frac{\partial}{\partial x_i}} \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_k} - \nabla_{\frac{\partial}{\partial x_j}} \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_k}.$$

Prove that

$$R_{ijk}^{\ \ \ l} = \frac{\partial \Gamma_{jk}^{\ l}}{\partial x_i} - \frac{\partial \Gamma_{ik}^{\ l}}{\partial x_j} + \Gamma_{jk}^{\ p}\Gamma_{ip}^{\ l} - \Gamma_{ik}^{\ p}\Gamma_{jp}^{\ l}.$$

Solution:

(a) Since $-\frac{\partial N}{\partial x_i} = \mathcal{W}\left(\frac{\partial \sigma}{\partial x_i}\right) = W^i_j \frac{\partial \sigma}{\partial x_i}$, then

$$h_{jl} = \left\langle \frac{\partial^2 \sigma}{\partial x_l \partial x_j}, N \right\rangle = \left\langle \frac{\partial \sigma}{\partial x_l}, -\frac{\partial N}{\partial x_i} \right\rangle = \left\langle \frac{\partial \sigma}{\partial x_l}, W^i_j \frac{\partial \sigma}{\partial x_i} \right\rangle = W^i_j g_{li}$$

Hence, $W^i_j = g^{il}h_{lj}$.

(b) Compute it directly

$$\frac{\partial}{\partial x_i} g_{jk} = \frac{\partial}{\partial x_i}\left\langle \frac{\partial \sigma}{\partial x_j}, \frac{\partial \sigma}{\partial x_k} \right\rangle = \left\langle \frac{\partial^2 \sigma}{\partial x_i \partial x_j}, \frac{\partial^2 \sigma}{\partial x_i \partial x_k} \right\rangle$$
$$\frac{\partial}{\partial x_j} g_{ki} = \cdots$$
$$\frac{\partial}{\partial x_k} g_{ji} = \cdots$$

Sum of above three equations, then $\frac{1}{2}\left(\frac{\partial}{\partial x^i} g_{jk} + \frac{\partial}{\partial x^j} g_{ki} - \frac{\partial}{\partial x^k} g_{ij}\right) = \left\langle \frac{\partial^2 \sigma}{\partial x^i \partial x^j}, \frac{\partial \sigma}{\partial x^k} \right\rangle$.

(c) Since $\frac{\partial^2 \sigma}{\partial x_i \partial x_j} = \Gamma_{ij}^{\ k} \frac{\partial \sigma}{\partial x_k} + h_{ij} N$, then

$$\left\langle \frac{\partial^2 \sigma}{\partial x_i \partial x_j}, \frac{\partial \sigma}{\partial x_l} \right\rangle = \left\langle \Gamma_{ij}^{\ k} \frac{\partial \sigma}{\partial x_k} + h_{ij} N, \frac{\partial \sigma}{\partial x_l} \right\rangle = \Gamma_{ij}^{\ k} g_{kl}$$

Moreover,

$$\left\langle \frac{\partial^2 \sigma}{\partial x^i \partial x^j}, \frac{\partial \sigma}{\partial x^k} \right\rangle = \frac{1}{2}\left(\frac{\partial}{\partial x^i} g_{jk} + \frac{\partial}{\partial x^j} g_{ki} - \frac{\partial}{\partial x^k} g_{ij}\right)$$

Hence,

$$\Gamma_{ij}^{\ k} = \frac{1}{2} g^{kl}\left(\frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{li}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l}\right)$$

(d) Compute directly,

$$\nabla_{\frac{\partial}{\partial x_i}} \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_k} = \nabla_{\frac{\partial}{\partial x_i}}\left(\Gamma_{jk}^{\ l} \frac{\partial}{\partial_l}\right) = \left(\frac{\partial}{\partial x_i} \Gamma_{jk}^{\ l}\right) \frac{\partial}{\partial x_l} + \Gamma_{jk}^{\ l} \Gamma_{il}^{\ p} \frac{\partial}{\partial x_p}$$
$$= \left(\frac{\partial}{\partial x_i} \Gamma_{jk}^{\ l} + \Gamma_{jk}^{\ p} \Gamma_{ip}^{\ l}\right) \frac{\partial}{\partial x_l}$$

Similarly, exchange i and j. Hence,

$$R_{ijk}^{\ \ \ l} \frac{\partial}{\partial x_l} = \nabla_{\frac{\partial}{\partial x_i}} \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_k} - \nabla_{\frac{\partial}{\partial x_j}} \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_k}$$
$$= \left(\frac{\partial}{\partial x_i} \Gamma_{jk}^{\ l} - \frac{\partial}{\partial x_j} \Gamma_{ik}^{\ l} - \Gamma_{ik}^{\ p} \Gamma_{jp}^{\ l} + \Gamma_{jk}^{\ p} \Gamma_{ip}^{\ l}\right) \frac{\partial}{\partial x_l}$$

(2) Let

$$g(x_1, \cdots, x_n) = \frac{\delta_{ij}}{x_n^2}$$

be the Riemannian metric on the set $\{(x_1, \cdots, x_n) : x_n > 0\}$.

(a) [5 pt] Show that Christoffel symbols are

$$\Gamma_{ij}^{\ m} = -\frac{1}{x_n}\left(\delta_{in}\delta_j^m + \delta_{jn}\delta_i^m - \delta_{ij}\delta_n^m\right)$$

(b) [10 pt] Compute the Riemannian curvature $R_{ijk}{}^{l}$ and the Ricci curvature. Show that $R_{jk} = -(n-1)g_{jk}$.

(c) [10 pt] Recall the geodesic $x_k'' + \sum_{ij} \Gamma_{ij}^{\ k} x_i' x_j'$, for all k. Given the half plane $\{(x_1, x_2) : x_n^2 > 0\}$ with the metric $g_{ij}(x_1, x_2) = \frac{\delta_{ij}}{x_2^2}$, find the geodesic of such manifold with $x_1(0) = 0$, $x_2(0) = 1$, $x_1'(0) = 1$ and $x_2'(0) = 0$.

Solution:

(a) We have $g_{ij} = \frac{\delta_{ij}}{x_n^2}$ and $g^{ij} = x_n^2 \delta^{ij}$, then

$$\partial_i g_{jl} = -\frac{1}{x_n^3}\delta_{in}\delta_{jl}.$$

Similarly, the results follow

$$\Gamma_{ij}^{\ m} = -\frac{1}{x_n}\left(\delta_{in}\delta_j^m + \delta_{jn}\delta_i^m - \delta_{ij}\delta_n^m\right) .$$

(b)

- **Riemannian curvature**

  First, do it by yourself,

$$\partial_i \Gamma_{jk}^{\ l} = \frac{1}{x_n^2}\left(\delta_{in}\delta_{jn}\delta_k^l + \delta_{in}\delta_{kn}\delta_j^l - \delta_{in}\delta_{jk}\delta_n^l\right) .$$

  Second, do it by yourself,

$$\Gamma_{ik}^{\ p}\Gamma_{jp}^{\ l} = \frac{1}{x_n^2}\left(2\delta_{in}\delta_{kn}\delta_j^l + \delta_{in}\delta_{jn}\delta_k^l + \delta_{jn}\delta_{kn}\delta_i^l - \delta_{in}\delta_{kj}\delta_n^l - \delta_{kn}\delta_{ij}\delta_n^l - \delta_{ik}\delta_j^l\right) .$$

  Hence,

$$R_{ijk}{}^{l} = \frac{1}{x_n^2}\left(\delta_{ik}\delta_j^l - \delta_{jk}\delta_i^l\right)$$

- **Ricci curvature**

$$R_{jk} = R_{ljk}{}^{l} = \frac{1}{x_n^2}\left(\delta_{lk}\delta_j^l - \delta_{jk}\delta_l^l\right) = \frac{1}{x_n^2}\left(\delta_{jk} - n\delta_{jk}\right) = \frac{-1}{x_n^2}(n-1)\delta_{jk}$$

- Hence, $R_{ij} = -(n-1)\frac{\delta_{ij}}{x_n^2} = -(n-1)g_{ij}$

Remark: It is different meaning of between superscripts and subscripts. The meaning of δ_l^l is sum over l from 1 to n, so $\delta_l^l = n$. However, δ_{ll} is (l, l)-component of matrix, so $\delta_{ll} = 1$.

(c) This is an example in my TA class. Please refer to the last page in my note. The following are Christoffel symbols.

$$\begin{array}{lllll} \Gamma_1: & \Gamma_{11}^1 = 0 & \Gamma_{11}^2 = \frac{1}{x_2} & \Gamma_{12}^1 = \frac{-1}{x_2} & \Gamma_{12}^2 = 0 \\ \Gamma_2: & \Gamma_{21}^1 = \frac{-1}{x_2} & \Gamma_{21}^2 = 0 & \Gamma_{22}^1 = 0 & \Gamma_{22}^2 = \frac{-1}{x_2} \end{array}$$

Hence,

$$\begin{cases} x_1'' - \frac{2}{x_2} x_1' x_2' = 0 \\ x_2'' + \frac{1}{x_2} x_1'^2 - \frac{1}{x_2} x_2'^2 = 0 \end{cases}$$

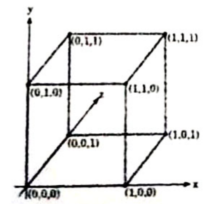
If $x_1' = 0$ then x_1 is constant. If $x_1' \neq 0$ then $x_1'' - \frac{2}{x_2} x_1' x_2' = \frac{d}{ds} \log\left(\frac{x_1'}{x_2^2}\right) = 0$ implies $x_1' = c x_2^2$. Let $x_1'^2 + x_2'^2 = x_2^2$ by the metric. You can solve two the first order ode equations by yourself, so $(x_1 - a)^2 + x_2^2 = \frac{1}{c^2}$. By initial condition, the geodesic is

$$\begin{cases} x = \cos(-s + \frac{\pi}{2}) \\ y = \sin(-s + \frac{\pi}{2}) \end{cases},$$

where $s \in [0, \infty)$.

Remark: This example is important because it breaks the **Euclid's fifth postulate**, which promote development of Riemannian geometry.

(3) $\boxed{15 \text{ pt}}$ A cube is a tri-valent graph as shown in figure.



Find the first, second fundamental forms, mean curvature and Gaussian curvature at $(0,0,0)$ of the cube given in the above graph. We choose the unit normal at each vertex to be the unit normal pointing inside the cube.

Solution: Start form $(1,0,0)$, and then let $e_1 = (-1,0,1)$ and $e_2 = (-1,1,0)$. Do it by yourself, the first fundamental form is

$$\mathcal{F}_{\mathrm{I}} = \begin{bmatrix} \langle e_1, e_1\rangle & \langle e_1, e_2\rangle \\ \langle e_2, e_1\rangle & \langle e_2, e_2\rangle \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Let N_0 is unit normal vector with inward-pointing at $(1,0,0)$, $N_0 = \frac{1}{\sqrt{3}}(-1,1,1)$. Moreover, $N_1 = \frac{1}{\sqrt{3}}(1,1,-1)$ and $N_2 = \frac{1}{\sqrt{3}}(1,-1,1)$. Hence, the second fundamental form is

$$\mathcal{F}_{\mathrm{II}} = \begin{bmatrix} \langle e_1, N_1 - N_0\rangle & \langle e_1, N_2 - N_0\rangle \\ \langle e_2, N_1 - N_0\rangle & \langle e_2, N_2 - N_0\rangle \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

Therefore, Weingarten map is

$$\mathcal{W} = \mathcal{F}_{\mathrm{I}}^{-1}\mathcal{F}_{\mathrm{II}} = \frac{1}{\sqrt{3}} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

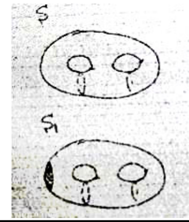
Thus, the mean curvature is

$$H = \frac{1}{2} \operatorname{tr} \mathcal{W} = \frac{2}{\sqrt{3}},$$

and the Gaussian curvature is

$$K = \det \mathcal{W} = \frac{4}{\sqrt{3}}.$$

(4) 10 pt Let S be the compact surface in the figure.



Let S_1 be the shaded region in the figure. Order these three numbers $\int_S K dA$, $\int_{S_1} K dA$ and $\int_{S \setminus S_1} K dA$ (Determine which number is the largest, second and the smallest.)

Solution: Do it by yourself!

(5) 10 pt Consider a non-compact surface with $K > 0$ that is topologically a cylinder. Prove that there cannot be two disjoint simple closed geodesics both going around the neck of the surface.

Solution: Suppose there are two closed simple curve γ_1 and γ_2 which are disjoint. That is, two curve enclose a topologically cylinder S', which is subset of original cylinder S, *i.e.* $S' \subseteq S$. Moreover, the boundary of S' is $\gamma_1 \cup \gamma_2$. Hence, by Gauss-Bonnet theorem with smooth boundary, *i.e.* $\phi_i = 0$,

$$2\pi\chi(S') = \int_{S'} K dA + \int_{\partial S} \kappa_g dA = \int_{S'} K dA > 0\,,$$

where $\kappa_g = 0$ and $K > 0$ on S. However, the Euler characteristic number of topologically cylinder is $\chi(S') = 0$, so the LHS equal to 0, which leads a contradiction to $0 > 0$. Therefore, there cannot be two disjoint simple closed geodesics both going around the neck of the surface.