# INTRODUCTION TO DIFFERENTIAL GEOMETRY MIDTERM EXAM SOLUTION (VERSION 3)

(1) (a) [20 pt] Let $S$ be smooth surface with unit normal vector $\mathbf{N}$. The Weingarten map $\mathcal{W}$ can be defined as $\mathcal{W}_{\sigma(u,v)}(a\sigma_u + b\sigma_v) = -a\mathbf{N}_u - b\mathbf{N}_v$. Prove that

$$K = \frac{LN - M^2}{EG - F^2} \quad \text{and} \quad H = \frac{1}{2}\frac{LG - 2MF + NE}{EG - F^2}.$$

(b) [25 pt] The surface is parametrized by

$$\sigma(u, v) = (\operatorname{sech}(u)\cos(v), \operatorname{sech}(u)\sin(v), u - \tanh(u)).$$

Find the mean curvature, Gaussian curvature, principal curvature and principle direction of the pseudosphere.

**Solution:**

(a) Let

$$\begin{cases} \mathcal{W}(\sigma_u) = -d\mathbf{N}(\sigma_u) = -\mathbf{N}_u = a\sigma_u + b\sigma_v \\ \mathcal{W}(\sigma_v) = -d\mathbf{N}(\sigma_v) = -\mathbf{N}_v = c\sigma_u + d\sigma_v \end{cases}.$$

Rewrite in matrix form

$$\begin{bmatrix} | & | \\ -\mathbf{N}_u & -\mathbf{N}_v \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ \sigma_u & \sigma_v \\ | & | \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

Multiply $[\sigma_u \ \sigma_v]^T$ both side,

$$\begin{bmatrix} - & \sigma_u & - \\ - & \sigma_v & - \end{bmatrix} \begin{bmatrix} | & | \\ -\mathbf{N}_u & -\mathbf{N}_v \\ | & | \end{bmatrix} = \begin{bmatrix} - & \sigma_u & - \\ - & \sigma_v & - \end{bmatrix} \begin{bmatrix} | & | \\ \sigma_u & \sigma_v \\ | & | \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}. \qquad [14\ pt]$$

Hence, $\mathcal{W} = \mathcal{F}_{\mathrm{I}}^{-1}\mathcal{F}_{\mathrm{II}}$. *[2 pt]*

Now, it's sufficient to compute $K$ and $H$.

$$K = \det \mathcal{W} = \frac{\det \mathcal{F}_{\mathrm{II}}}{\det \mathcal{F}_{\mathrm{I}}} = \frac{LN - M^2}{EG - F^2}, \qquad [2\ pt]$$

and

$$H = \frac{1}{2}\operatorname{tr}\mathcal{W} = \frac{1}{2}\operatorname{tr}\left(\frac{1}{EG - F^2}\begin{bmatrix} G & -F \\ -F & E \end{bmatrix}\begin{bmatrix} L & M \\ M & N \end{bmatrix}\right) = \frac{1}{2}\frac{LG - 2MF + NE}{EG - F^2}. \qquad [2\ pt]$$

(b) The following can be computed directly

$$\sigma_u = \left(-\operatorname{sech} u \tanh u \cos v,\ -\operatorname{sech} u \tanh u \sin v,\ \tanh^2 u\right)$$

$$\sigma_v = (-\operatorname{sech} u \sin v,\ \operatorname{sech} u \cos v,\ 0)\ .$$

Hence,

$$\begin{aligned} E &= \langle \sigma_u, \sigma_u \rangle = \tanh^2 u & L &= \langle \sigma_{uu}, N \rangle = -\operatorname{sech} u \tanh u \\ F &= \langle \sigma_u, \sigma_v \rangle = 0 & M &= \langle \sigma_{uv}, N \rangle = 0 \\ G &= \langle \sigma_v, \sigma_v \rangle = \operatorname{sech}^2 u & N &= \langle \sigma_{vv}, N \rangle = \operatorname{sech} u \tanh u\,. \end{aligned}$$

Therefore, the first and second fundamental form is

$$\mathcal{F}_{\mathrm{I}} = \begin{bmatrix} \tanh^2 u & 0 \\ 0 & \operatorname{sech}^2 u \end{bmatrix} \quad \mathcal{F}_{\mathrm{II}} = \begin{bmatrix} -\operatorname{sech} u \tanh u & 0 \\ 0 & \operatorname{sech} u \tanh u \end{bmatrix}. \qquad \textit{[9 pt]}$$

(i) **8 pt** It's sufficient to compute principal curvature and direction, $\det(\mathcal{F}_{\mathrm{I}}^{-1}\mathcal{F}_{\mathrm{II}} - \kappa I) = \det(\mathcal{F}_{\mathrm{II}} - \kappa \mathcal{F}_{\mathrm{I}}) = 0$. Thus, the principal curvature is

$$\kappa_1 = -\frac{\operatorname{sech} u}{\tanh u} = -\sinh u \quad \text{or} \quad \kappa_2 = \frac{\tanh u}{\operatorname{sech} u} = \operatorname{csch} u\,.$$

Then,

$$(\mathcal{F}_{\mathrm{I}}^{-1}\mathcal{F}_{\mathrm{II}} - \kappa_1 I)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad (\mathcal{F}_{\mathrm{I}}^{-1}\mathcal{F}_{\mathrm{II}} - \kappa_2 I)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence,

principal curvature $\kappa_1 = -\sinh u$ corresponding to principal vector $\sigma_u$ *[4 pt]*

principal curvature $\kappa_1 = \operatorname{csch} u$ corresponding to principal vector $\sigma_v$ . *[4 pt]*

(ii) **8 pt** Moreover, the

$$K = \kappa_1 \kappa_2 \qquad \textit{[1 pt]}$$

$$= -1 \qquad \textit{[3 pt]}$$

$$H = \frac{1}{2}(\kappa_1 + \kappa_2) \qquad \textit{[1 pt]}$$

$$= \frac{\operatorname{sech}^2 u - \tanh^2 u}{2 \operatorname{sech} u \tanh u} = \frac{1}{2}(-\sinh u + \operatorname{csch} u)\,. \qquad \textit{[3 pt]}$$

(2) (a) [5 pt] Let $S$ be a compact surface without boundary and unit vector $A \in \mathbb{R}^3$. Let $f(p) = \langle p, A \rangle$ for $p \in S$. Prove that a point $p^* \in S$ is a critical point of $f$ if and only if the normal line of $S$ at $p^*$ parallel to $A$.

(b) [5 pt] Show that there is a critical point of $S$ whose normal line is parallel to vector $A$.

(c) [5 pt] $p^*$ is global maximum of the function $f$. Determine the sign of the Gaussian curvature.

(d) [5 pt] Is it possible to have a compact surface whose Gaussian curvature is negative?

**Solution:**

(a) $\Rightarrow$) Given any $v \in T_{p^*}S$, define a curve $\gamma(t)$ on $S$ with $\gamma(0) = P^*$ and $\gamma'(0) = v$. Since $\frac{d}{dt}\Big|_{t=0} f(\gamma(t)) = Df_{p^*}(v)$,

$$Df_{p^*}(v) = \frac{d}{dt}\bigg|_{t=0} \langle \gamma(t), A \rangle = \langle \gamma'(0), A \rangle = \langle v, A \rangle .$$

Now $p^*$ is a critical point. It implies $\langle v, A \rangle = 0$ for all $v \in T_{p^*}S$ and $A$ is parallel to normal vector at $p^*$. So the normal line thru $p^*$ is parallel to the vector $A$.

$\Leftarrow$) Since the normal line of $S$ at $p*$ parallel to $A$, then $\langle v, A \rangle = 0$. This implies

$$Df_{p^*}(v) = \frac{d}{dt}\bigg|_{t=0} \langle \gamma(t), A \rangle = \langle v, A \rangle = 0 .$$

Thus, $p^*$ is critical point.

(b) Since $S$ is compact surface without boundary, $f$ can attain maximum and minimum value on $S$. By Problem (2)(a)($\Rightarrow$), exist a critical point $p^*$ such that the normal line at $p^*$ parallel to $A$.

(c) Let the curve $\gamma$ be a unit speed curve with $\gamma'(0) = v$. Since $f$ has global maximum at $p^*$, then $p^*$ is a critial point of $f$ and $\frac{d^2}{dt^2}\Big|_{t=0} f(\gamma(t)) = \langle A, \gamma''(p^*)\rangle \leq 0$. Since $p^*$ is critical point, a vector $A$ is parallel to normal vector $\mathbf{N}$ such that $A = \mathbf{N}(p^*)$ or $A = -\mathbf{N}(p^*)$. Hence, either $\langle \mathbf{N}(p^*), \gamma''(p^*)\rangle \leq 0$ or $\langle \mathbf{N}(p^*), \gamma''(p^*)\rangle \geq 0$ will happen. Let principal curvature $\kappa_1 \geq \kappa_2$ at $p^*$, with principal direction $u_1$ and $u_2$.

First consider $\langle \mathbf{N}(p^*), \gamma''(p^*)\rangle = \langle \mathcal{W}_{p^*}(v), v\rangle \leq 0$. By the definition of principal curvature,

$$\kappa_1 = \max_{\|v\|=1} \langle \mathcal{W}_{p^*}(v), v\rangle \leq 0 \quad \text{and} \quad \kappa_2 = \min_{\|v\|=1} \langle \mathcal{W}_{p^*}(v), v\rangle \leq 0$$

Hence, Gaussian curvature $K = \kappa_1\kappa_2 \geq 0$. Similarly, consider $\langle \mathbf{N}(p^*), \gamma''(p^*)\rangle = \langle \mathcal{W}_{p^*}(v), v\rangle \geq 0$. By the definition of principal curvature,

$$\kappa_1 = \max_{\|v\|=1} \langle \mathcal{W}_{p^*}(v), v\rangle \geq 0 \quad \text{and} \quad \kappa_2 = \min_{\|v\|=1} \langle \mathcal{W}_{p^*}(v), v\rangle \geq 0$$

Hence, Gaussian curvature $K = \kappa_1\kappa_2 \geq 0$. Therefore, Gaussian curvature $K \geq 0$ at global maximum point $p^*$.

(d) Since a compact surface $S$ without boundary always can be attained critical point of $f$ on $S$. By Problem (2)(c), the argument of minimum point is similar. Hence, exist at least a point such that Gaussian curvature is non-negative.

(3) Define the third fundamental form of Gauss map $\mathbf{N}$ by

$$\mathcal{F}_{\mathrm{III}} = \begin{bmatrix} \mathbf{N}_u \cdot \mathbf{N}_u & \mathbf{N}_u \cdot \mathbf{N}_v \\ \mathbf{N}_v \cdot \mathbf{N}_u & \mathbf{N}_v \cdot \mathbf{N}_v \end{bmatrix}$$

(a) $\boxed{10 \text{ pt}}$ Prove that $\mathcal{F}_{\mathrm{III}} = \mathcal{F}_{\mathrm{II}}\mathcal{F}_{\mathrm{I}}^{-1}\mathcal{F}_{\mathrm{II}}$.

(b) $\boxed{10 \text{ pt}}$ Prove that $\mathcal{F}_{\mathrm{III}} - 2H\mathcal{F}_{\mathrm{II}} + K\mathcal{F}_{\mathrm{I}} = 0$.

**Solution:**

(a) By the similar work in Problem 1(a),

$$\begin{cases} -\mathbf{N}_u &= a\sigma_u + b\sigma_v \\ -\mathbf{N}_v &= c\sigma_u + d\sigma_v \end{cases}$$

Hence, $\langle \mathbf{N}_u, \mathbf{N}_u \rangle = \langle -a\sigma_u - b\sigma_v, \mathbf{N}_u \rangle = -a\langle \sigma_u, \mathbf{N}_u \rangle - b\langle \sigma_v, \mathbf{N}_u \rangle = aL + bM$. Doing it by yourself, the result

$$\mathcal{F}_{\mathrm{III}} = \begin{bmatrix} L & M \\ M & N \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \mathcal{F}_{\mathrm{II}}\mathcal{F}_{\mathrm{I}}^{-1}\mathcal{F}_{\mathrm{II}}$$

can be follows.

**Another Method 1.** According to Problem (1),

$$\begin{bmatrix} | & | \\ -\mathbf{N}_u & -\mathbf{N}_v \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ \sigma_u & \sigma_v \\ | & | \end{bmatrix} \mathcal{F}_{\mathrm{I}}^{-1}\mathcal{F}_{\mathrm{II}}\,.$$

Therefore,

$$\begin{aligned} \mathcal{F}_{\mathrm{III}} &= \begin{bmatrix} — & -\mathbf{N}_u & — \\ — & -\mathbf{N}_v & — \end{bmatrix} \begin{bmatrix} | & | \\ -\mathbf{N}_u & -\mathbf{N}_v \\ | & | \end{bmatrix} \\ &= \begin{bmatrix} — & -\mathbf{N}_u & — \\ — & -\mathbf{N}_v & — \end{bmatrix} \begin{bmatrix} | & | \\ \sigma_u & \sigma_v \\ | & | \end{bmatrix} \mathcal{F}_{\mathrm{I}}^{-1}\mathcal{F}_{\mathrm{II}} = \mathcal{F}_{\mathrm{II}}\mathcal{F}_{\mathrm{I}}^{-1}\mathcal{F}_{\mathrm{II}}\,. \end{aligned}$$

(a) **Another Method 2.** Push the vector to $sp(\sigma_u, \sigma_v)$. Write $-N_u = a\sigma_u + b\sigma_v$ and $-N_v = c\sigma_u + d\sigma_v$.

$$\begin{bmatrix} — & -\mathbf{N}_u & — \\ — & -\mathbf{N}_v & — \end{bmatrix} \begin{bmatrix} | & | \\ -\mathbf{N}_u & -\mathbf{N}_v \\ | & | \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \mathcal{F}_{\mathrm{II}} \mathcal{F}_{\mathrm{I}}^{-1} \mathcal{F}_{\mathrm{II}} \,.$$

**Another Method 3.** Similar to method 2 and **note the $\mathbf{N}_u$ and $\mathbf{N}_u$ decomposition**. Compute directly first,

$$\mathcal{F}_{\mathrm{II}} \mathcal{F}_{\mathrm{I}}^{-1} \mathcal{F}_{\mathrm{II}} = \frac{1}{EG - F^2} \begin{bmatrix} L^2G - LMF + M^2E & \spadesuit \\ \bigstar & \clubsuit \end{bmatrix} \qquad [\textit{2 pt}]$$

Then,

$$L^2G - LMF + M^2E = \langle \sigma_u, \mathbf{N}_u \rangle^2 G - 2\langle \sigma_u, \mathbf{N}_u \rangle \langle \sigma_v, \mathbf{N}_u \rangle F + \langle \sigma_v, \mathbf{N}_u \rangle^2 E \,. \qquad [\textit{2 pt}]$$

Since $\mathbf{N}_u = \langle \frac{\sigma_u}{\|\sigma_u\|}, \mathbf{N}_u \rangle \frac{\sigma_u}{\|\sigma_u\|}$, then

$$\langle \mathbf{N}_u, \mathbf{N}_u \rangle = \left( \langle \sigma_u, \mathbf{N}_u \rangle^2 \frac{1}{E} + 2\langle \sigma_u, \mathbf{N}_u \rangle \langle \sigma_v, \mathbf{N}_u \rangle \frac{F}{EG} + \langle \sigma_v, \mathbf{N}_u \rangle^2 \frac{1}{G} \right) \qquad [\textit{5 pt}]$$

Hence,

$$(EG - F^2) \langle \mathbf{N}_u, \mathbf{N}_u \rangle = L^2G - LMF + M^2E \,.$$

Doing it by yourself, $\mathcal{F}_{\mathrm{III}} = \mathcal{F}_{\mathrm{II}} \mathcal{F}_{\mathrm{I}}^{-1} \mathcal{F}_{\mathrm{II}}$ can be follows. [*1 pt*]

(b) **Method 1.** Recall Cayley-Hamilton theorem, the matrix $A$ is satisfied its characteristic polynomial $p(A) = 0$, where $p(\lambda) = \lambda^2 - \mathrm{tr}(A)\lambda + \det(A)$ in $2 \times 2$ case.

Hence, the characteristic polynomial of $\mathcal{W}$ is $p(\lambda) = \lambda^2 - 2H\lambda + K$. Plug $\mathcal{W} = \mathcal{F}_{\mathrm{I}}^{-1}\mathcal{F}_{\mathrm{II}}$ into $p$. Thus,

$$p(\mathcal{W}) = \mathcal{F}_{\mathrm{I}}^{-1}\mathcal{F}_{\mathrm{II}}\mathcal{F}_{\mathrm{I}}^{-1}\mathcal{F}_{\mathrm{II}} - 2H\mathcal{F}_{\mathrm{I}}^{-1}\mathcal{F}_{\mathrm{II}} + K = 0\,.$$

Multiply $\mathcal{F}_{\mathrm{I}}$ on both side from left,

$$\mathcal{F}_{\mathrm{III}} - 2H\mathcal{F}_{\mathrm{II}} + K\mathcal{F}_{\mathrm{I}} = 0\,.$$

**Method 2.** Similar to method I. Since $\kappa_1$ and $\kappa_2$ is eigenvalue of $\mathcal{W}$, consider

$$(\mathcal{W} - \kappa_1 I)(\mathcal{W} - \kappa_2 I) = \mathbf{O}$$

Then, $\mathcal{W}^2 - (\kappa_1 + \kappa_2)\mathcal{W} + \kappa_1\kappa_2 = \mathbf{O}$. By the same work in method I, $\mathcal{F}_{\mathrm{III}} - 2H\mathcal{F}_{\mathrm{II}} + K\mathcal{F}_{\mathrm{I}} = 0$ can be gotten.

**Method 3.** Compute directly. Plug $K = \frac{LN-M^2}{EG-F^2}$ and $H = \frac{1}{2}\frac{LG-2MF+NE}{EG-F^2}$ into equation,

$$2H\mathcal{F}_{\mathrm{II}} - K\mathcal{F}_{\mathrm{I}} = \frac{1}{EG-F^2}\begin{bmatrix} L^2G - 2LME + LNF - LNE + M^2E & \spadesuit' \\ \bigstar' & \clubsuit' \end{bmatrix} \qquad [4\ pt]$$

Doing it by yourself, then $\mathcal{F}_{\mathrm{III}} - 2H\mathcal{F}_{\mathrm{II}} + K\mathcal{F}_{\mathrm{I}} = 0$ can be gotten. *[6 pt]*

(4) $\boxed{\text{15 pt}}$ Let $\gamma$ is a curve with unit speed. The normal line to $\gamma$ with direction $\gamma''$. Suppose all normal line to $\gamma$ pass through a fixed point. What can you say about the curve?

**Solution:** Let $\{t, n, b\}$ be Frenet frame. Define the normal line as $\gamma(s) + \bar{\lambda}(s)\gamma''(s) = p_0$, which passes the fixed point $p_0$. Since $\gamma'' = t' = \kappa n$, above equation can be rewrited as

$$\gamma(s) + \lambda(s)n(s) = p_0 \,.$$

Note that $n$ is a vector between $p_0$ and $\gamma(s)$ and $|\lambda|$ is distance between $p_0$ and $\gamma(s)$. Before calculation, claim $\lambda$ is smooth. Clearly, $\lambda$ can be written as $\lambda = \langle \gamma - p_0, n\rangle$. Since $\gamma$ is smooth, $n = \frac{1}{\kappa}t$ is smooth where $\kappa = \|\gamma''(s)\|$ is smooth. Derivative both side,

$$t + \lambda' n + \lambda n' = 0 \,,$$

so $t + \lambda' n + \lambda(-\kappa t + \tau b) = 0$. Since $\{t, n, b\}$ is linear independent,

$$\begin{cases} \tau = 0 \\ \lambda' = 0 \\ 1 - \lambda\kappa = 0 \end{cases}$$

Thus, since $\tau = 0$, the curve lies on plane. Since $\lambda' = 0$, the distance between curve and fixed point $p_0$ is constant. Moreover, $\kappa = \frac{1}{\lambda}$ is constant. Therefore, the curve $\gamma$ is lie on a circle $S^1$.