A NOTE OF GEOMETRIC INDEX CONVENTION (VERSION 1)

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1. Why superscripts and subscripts

In **mathematics** convention, vectors are denoted by lower indices as ∂_i , and covectors are denoted by upper indices as dx^i. Hence, a covector eat a vector and spit a value, which can be written as

$$dx^i(\partial_j) = \delta^i_j \,.$$

Hence, the operator A^i_j could be regarded as a matrix that map a vector (covector) to another vector (covector). For example, operator A map vector X_i to $A^i_j X_i$, where X_i is a vector, so $A^i_j X_i$ is the new vector.

However, in **physics** convention[1], they write the components, so vectors are denoted by upper indices as $a^i\partial_i$, and covectors are denoted by upper indices as $\omega_i dx^i$.

Remark that when we write metric g_{ij} is a 2-form (covector), we use the idea of physics convention. That is, $g_{ij}dx^i \otimes dx^j$.

2. Christoffel symbols

Usually, we write Christoffel symbols Γ^k_{ij} but it's not a good type. Let's look inside the definition of Christoffel symbols. Starting from covariant derivative, we know

$$\nabla_i(a^j\partial_j) = (\partial_i a^k + \Gamma^k_{ij})\partial_k \,,$$

where a vector $X = a^i\partial_i$. Hence, Christoffel symbols are a map, which is decided by eating a vector i, and then the matrix $(\Gamma_i)^k_j$ maps a vector j to another vector. Precisely, $(\Gamma^k_{ij})_{jk}$ is a matrix-valued 1-form,

$$(\Gamma^k_{ij})_{jk} \in \Gamma(T^*M \otimes \mathfrak{gl}(n, \mathbb{R}))\,.$$

Therefore, the Christoffel symbols should be written as

$$\Gamma_{ij}{}^k$$

which align k with j. Thus, we usually write

$$\nabla_i = \partial_i + \Gamma_i \,.$$

[1]Please refer to Carroll [4].

If we write Γ^k_{ij}, it means eat a vector j, and then $(\Gamma_j)^k_i$ maps a vector i to another vector.

However, $\Gamma_{ij}{}^k = \Gamma_{ji}{}^k$ if torsion free, so it doesn't matter whether Γ^k_{ij} or $\Gamma_{ij}{}^k$ is written. But the next section is important, and actually, **every books has their own definition, so be careful about that**.

3. Riemannian curvature

The definition of Riemannian curvature is

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

where X, Y, Z is tangent vector. Wrire in coordinate form,

$$R(\partial_i, \partial_j)\partial_k = R_{ijk}{}^l \partial_l \,.$$

THat is, it eat two vectors i, j and then $R(\partial_i, \partial_j)$ or $(R_{ij})^l_k$ map a vector k to another vector. Precisely, $(R_{ijk}{}^l)_{kl}$ is a matrix-valued 2-form,

$$(R_{ijk}{}^l)_{kl} \in \Gamma(T^*M \otimes T^*M \otimes \mathfrak{gl}(n, \mathbb{R}))\,.$$

Hence, in John Lee [2], the index k and l are aligned and placed last,

$$R_{ijk}{}^l \partial_l = R(\partial_i, \partial_j)\partial_k \,.$$

On the other hand, in Mantredo do Carmo [1], the index k and l are aligned and placed first,

$$R^l_{kij} \partial_l = R(\partial_i, \partial_j)\partial_k \,.$$

Therefore, **remind yourself all the time**

$$R^l_{ijk} \neq R_{ijk}{}^l \,.$$

By the way, I **memorize** the curvature formula as $R_{ijk}{}^l = \partial_i \Gamma_j - \partial_j \Gamma_i - \Gamma_i \Gamma_j + \Gamma_j \Gamma_i$ and filled in k, l by the same position, $R_{ijk}{}^l = \partial_i \Gamma_{jk}{}^l - \partial_j \Gamma_{ik}{}^l - \Gamma_{ik}{}^\xi \Gamma_{j\xi}{}^l + \Gamma_{jk}{}^\xi \Gamma_{i\xi}{}^l$

References

[1] Mantredo do Carmo, *Riemannian Geometry.*

[2] John Lee, *Riemannian Manifolds: An Introduction to Curvature.*

[3] Jurgen Jost, *Riemannian Geometry and Geometric Analysis.*

[4] Sean Carroll, *Spacetime and Geometry: An Introduction to General Relativity.*