# A Numerical Study of Einstein Scalar Field Equationsin Spherically Symmetric Spacetime

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# Statement of the Problem (1)

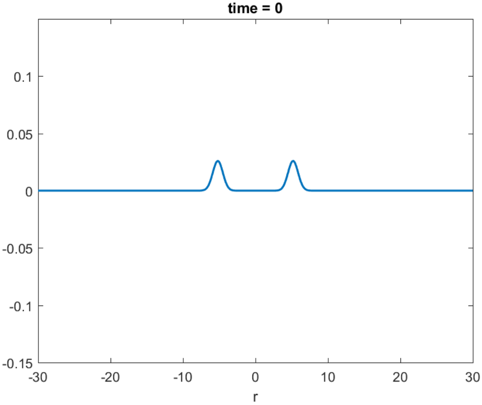
Einstein equations $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}$ are highly nonlinear PDEs.

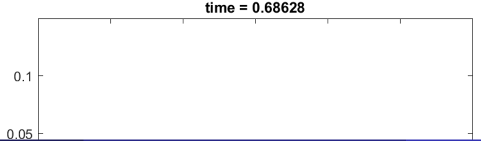
## Goal

Consider the Einstein scalar field equations $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi(\psi_\mu \psi_\nu - \frac{1}{2}|\nabla\psi|^2 g_{\mu\nu})$ or $R_{\mu\nu} = 8\pi\psi_\mu\psi_\nu$. In the spherical symmetric case, given initial scalar field, rewrite Einstein scalar equations to solve for the metric tensor.

# Statement of the Problem (2)

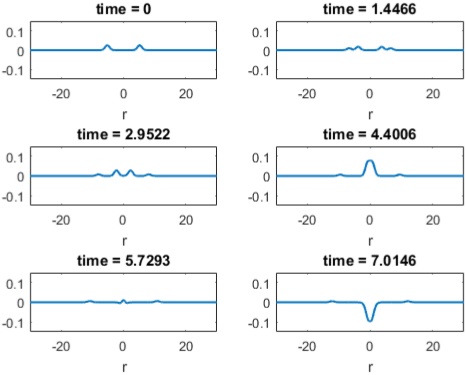
# Scalar field



# Statement of the Problem (3)

Given the initial scalar field, distinguish whether the incoming wave will be trapped near origin to form a back hole or not.

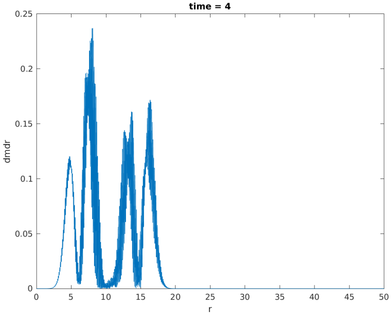



The blue line is the scalar field.

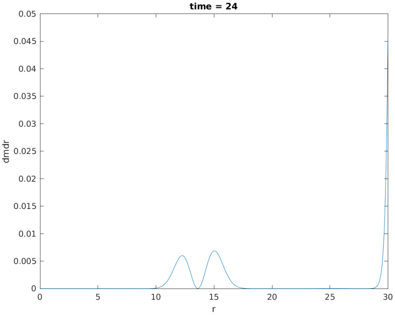
# Statement of the Problem (4)

This thesis not only confirms the simulative results of Choptuik, *i.e.* critical phenomena, but also summarize useful numerical techniques.

# It is not so easy as you might think.




(a) Oscilation if one uses the naive numerical schemes

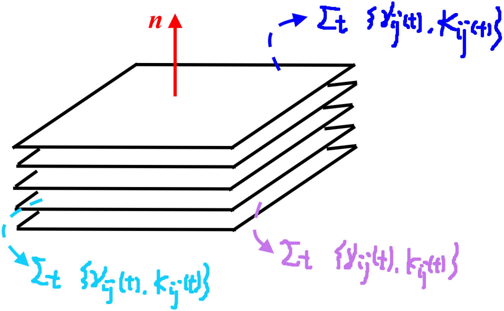



(b) The energy of solution is increasing if ones does not use boundary condition.

# Outline

# ADM split

According to Arnowitt *et al.* [2], 4-dimension spacetime $\bar{M}$ split into time slices $\Sigma_t$.



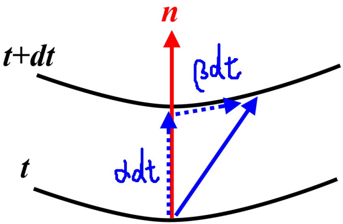

The pair 3-tensor $\{\gamma_{ij}, K_{ij}\}$ form the fundamental dynamical variable of the evolution of the spacetime.

# Lapse function and shift vector

The metric can be written as

$$g = g_{\mu\nu} dx^\mu dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$
$$= -(\alpha^2 - \gamma_{ij}\beta^i\beta^j) dt^2 + 2\gamma_{ij}\beta^j dt dx^i + \gamma_{ij} dx^i dx^j \,.$$

- Lapse function $\alpha$ measure "Proper time/Coordinate time".
- Shift vector $\beta$ measure how the coordinates move.



# Coordinates conditions

Einstein equations say nothing about gauge variables.

**Goal**

In particular, black holes may contain singularities. Coordinates conditions must avoid singularities that appear in the evolution.

There are two steps:

- choosing a time slicing condition for $\alpha$
- choosing a spatial gauge for $\beta^i$ .

# Polar-areal slicing condition

The spherical coordinates $\{r, \theta, \phi\}$ is chosen. Since spherical symmetry, $\beta^\theta = \beta^\phi = 0$ and denote $\beta = \beta^r$.

The metric can be written as

$$g = -(\alpha^2 - a^2\beta^2)dt^2 + 2a^2\beta\, dt dr + a^2 dr^2 + r^2 b^2 d\Omega^2 \,.$$

**Areal-Polar slicing condition**

$K = K_r^r$ *i.e.* $K_\theta^\theta + K_\phi^\phi = 0$

Since spherical symmetry, $K_\theta^\theta = K_\phi^\phi = 0$. Thus, $\beta = 0$, $b = 1$. Hence, the metric can be written as

$$\boxed{g = -\alpha^2 dt^2 + a^2 dr^2 + r^2 d\Omega^2 \,.}$$

# Auxillary variables

Given a massless scalar field $\psi$, it has to satisfy

$$T_{\mu\nu} = \nabla_\mu \psi \nabla_\nu \psi - \frac{1}{2} g_{\mu\nu} \nabla^\alpha \psi \nabla_\alpha \psi \, .$$

Define the auxillary variable for scalar field $\psi$ as following,

$$\Phi(t, r) := \partial_r \psi$$
$$\Pi(t, r) := \frac{a}{\alpha} \partial_t \psi \ .$$

# Solve metric on time slices

- Since $K^2 = K^{ij}K_{ij}$, the **Hamiltonian constraint** becomes $R = 16\pi\rho = 16\pi(\Phi^2 + \Pi^2)/(2a^2)$. Then,

$$\boxed{\frac{1}{a}\frac{\partial a}{\partial r} + \frac{a^2 - 1}{2r} - 2\pi r(\Phi^2 + \Pi^2) = 0}.$$

- Since $K_{\theta\theta} = 0$, the slice condition $\frac{\partial K_{\theta\theta}}{\partial t} = 0$,

$$\boxed{\frac{1}{\alpha}\frac{\partial \alpha}{\partial r} - \frac{1}{a}\frac{\partial a}{\partial r} - \frac{a^2 - 1}{r} = 0}.$$

# Evolution of scalar field

- Scalar field $\psi$ must satisfy $\Box\psi = 0$, so
  $\nabla^\mu \nabla_\mu \psi = g^{\mu\alpha}(\partial_\alpha \nabla_\mu - \Gamma^\xi_{\alpha\mu}\nabla_\xi)\psi$.

$$\boxed{\frac{\partial \Pi}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\alpha}{a}\Phi\right)}.$$

- Scalar field $\psi$ is smooth, $\frac{\partial}{\partial r}\frac{\partial \psi}{\partial t} = \frac{\partial}{\partial t}\frac{\partial \psi}{\partial r}$, so

$$\boxed{\frac{\partial \Phi}{\partial t} = \frac{\partial}{\partial r}\left(\frac{\alpha}{a}\Pi\right)}.$$

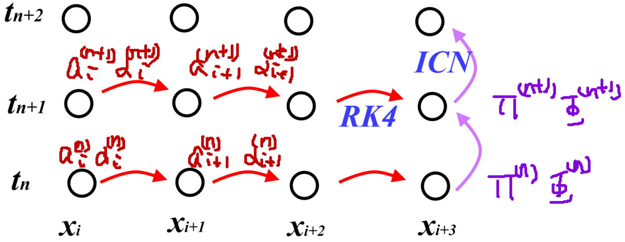
# Evolution system

Hence, the evolution system is as

$$\begin{cases} \frac{1}{a}\frac{\partial a}{\partial r} + \frac{a^2-1}{2r} - 2\pi r(\Phi^2 + \Pi^2) = 0 \\ \frac{1}{\alpha}\frac{\partial \alpha}{\partial r} - \frac{1}{a}\frac{\partial a}{\partial r} - \frac{a^2-1}{r} = 0 \\ \frac{\partial \Pi}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\alpha}{a}\Phi\right) \\ \frac{\partial \Phi}{\partial t} = \frac{\partial}{\partial r}\left(r^2 \frac{\alpha}{a}\Phi\right) \end{cases}$$

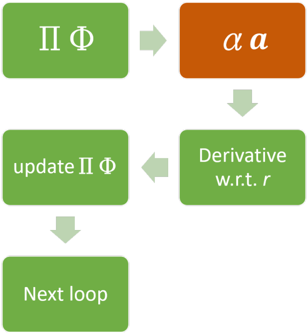
Boundary condition is

$$\begin{cases} a(r=0) = 1 \\ \alpha(r=r_N) = \frac{1}{a(r=r_N)} \\ \Phi(r=0) = 0 \\ \partial_t \Phi(r=0) = 0 \end{cases}$$



# Evolution system

$$\begin{cases} \frac{1}{a}\frac{\partial a}{\partial r} + \frac{a^2-1}{2r} - 2\pi r(\Phi^2 + \Pi^2) = 0 \\ \frac{1}{\alpha}\frac{\partial \alpha}{\partial r} - \frac{1}{a}\frac{\partial a}{\partial r} - \frac{a^2-1}{r} = 0 \\ \frac{\partial \Pi}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\alpha}{a}\Phi\right) \\ \frac{\partial \Phi}{\partial t} = \frac{\partial}{\partial r}\left(r^2\frac{\alpha}{a}\Phi\right) \end{cases}$$



# Outline

1. Introduction
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3. Locating BH Horizon
   - Apparent horizon
   - Mass aspect function
4. Numerical Theoretical Analysis
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# Horizon

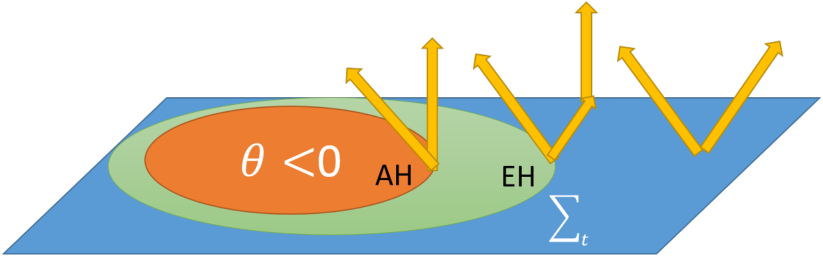
There are two different types of horizon of black holes.

- Event horizon (EH) is outgoing future-direction null geodesic neither reach infinity nor fall toward the center of singularity.
- An apparent horizon (AH) is defined as the divergence of the outgoing light rays vanish on a closed 2-surface in $\Sigma_t$.

## Remark

- Note that the EH is global property so it is difficult to simulate numerically.
- According to Hawking's paper, AH must inside EH if AH exists on $\Sigma_t$.

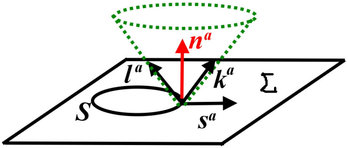
# Relation between EH and AH




Orange region is trapped region.

# AH equation

Consider a smooth, closed 2-dimension surface $S$ in spatial slice $\Sigma_t$.




The expansion of the outgoing null vector $k^i$ normal to $S$ is defined by

$$\Theta = \nabla_\mu k^\mu \, .$$

The AH equation is

$$\Theta = \gamma^{ij} D_i s_j - K + s^i s^j K_{ij} = 0 \, .$$

# Singularity avoidence (1)

Consider the spatial metric in spherical symmetric coordinate

$$\gamma = a^2 dr^2 + r^2 b^2 d\Omega^2 \,.$$

Choose the vector $s^i = (\frac{1}{a}, 0, 0)$ with $\gamma_{ij} s^i s^j = 1$. Then,

$$D_i s^i = \frac{1}{\sqrt{\gamma}} \partial_i (\sqrt{\gamma} s^i) = \frac{1}{ar^2 b^2} \partial_r (r^2 b^2) = \frac{2}{arb} \partial_r (rb) \,.$$

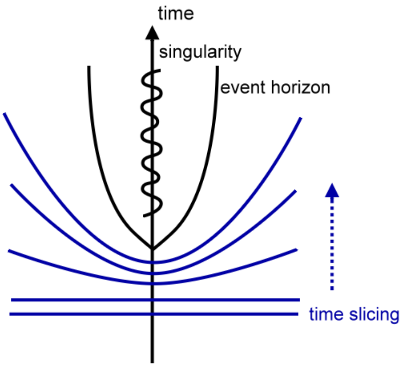
Hence,

$$\partial_r (rb) = arbK_\theta^\theta \,.$$

However, the areal-polar slicing condition is chosen, *i.e.* $b = 1$ and $K_\theta^\theta = 0$. Thus the equation above always does not hold, *i.e.* $1 \neq 0$.

# Singularity avoidence (2)

Therefore, the areal-polar slicing condition is not only singularity avoidance but also preventing the apparent horizon.



# No AH in polar areal slicing condition

By above subsection, polar-areal slicing condition can not cross the apparent horizon.

**Question:** How to locate black holes?

# Mass aspect function

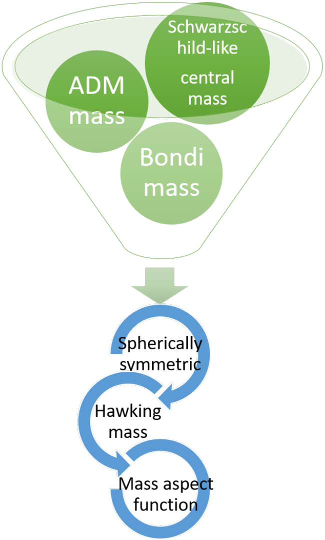
In the Schwarzschild-like metric,

$$\left(1 - \frac{2m}{r}\right)^{-1} = a^2 \,.$$

From Bardeen and Piran's paper, the mass aspect function can be defined as follows

$$m(r,t) = \frac{1}{2} r (1 - \frac{1}{a^2}) \,.$$

# Inheriting Hawking mass advantage



# Advantage of Hawking mass

- Compute ADM mass,

$$\lim_{r \to \infty} m_H(r) = M_{ADM} \,,$$

  on a spatial slice $\Sigma_t$ where $M_{ADM}$ is ADM mass.
- a black hole forms when $a \to \infty$ i.e $\frac{2m}{r} \to 1$. It is located at areal radius $r = R_{BH}$ where $m$ is mass aspect function.
- Therefore, the mass of final black hole can be computed by $M_{BH} = R_{BH}/2$.

# Outline

# Characteristic analysis of scalar field (1)

Consider the evolution of scalar field

$$\partial_t \Pi = \frac{\alpha}{a}\partial_r \Phi + l.o.$$
$$\partial_t \Phi = \frac{\alpha}{a}\partial_r \Pi + l.o..$$

where $l.o.$ denote the lower order term. Rewrite in matrix form $u_t + Au_x = 0$, where vector $u = [\Pi, \Phi]^T$ and

$$A = \begin{bmatrix} 0 & -\frac{\alpha}{a} \\ -\frac{\alpha}{a} & 0 \end{bmatrix}.$$

# Characteristic analysis of scalar field (2)

The eigenvector of $A$ is $[1 \quad -1]^T$ w.r.t eigenvalue $\frac{\alpha}{a}$.
The eigenvector of $A$ is $[1 \quad 1]^T$ w.r.t eigenvalue $-\frac{\alpha}{a}$.
Hence, if $\frac{\alpha}{a} \neq 0$, then there are two different eigenvalue. Exist the matrix

$$R = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

such that

$$w_t + \begin{bmatrix} \frac{\alpha}{a} & 0 \\ 0 & -\frac{\alpha}{a} \end{bmatrix} w_x = 0\,,$$

where

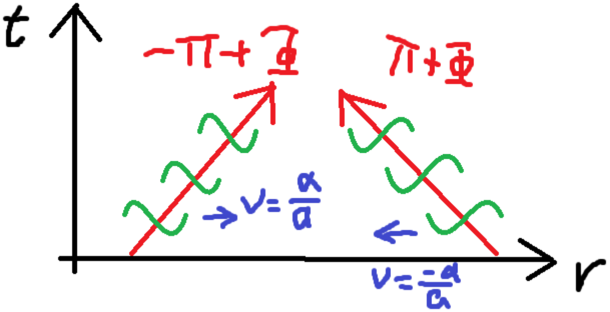
$$w = R^{-1} u = \begin{bmatrix} -\Pi + \Phi \\ \Pi + \Phi \end{bmatrix}$$

is called eigenfield.

# Well-posed

The evolution equation can be **decoupled** into two eigenfield.

Then, the hyperbolic system can be shown to be **well-posed**.



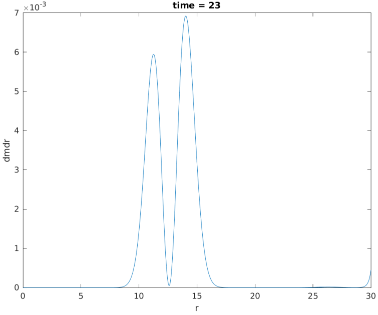
# Numerical Radiation Boundary Conditions

**Question**

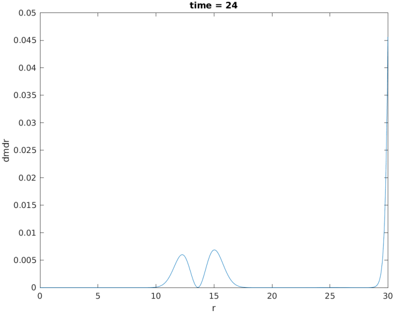
The computational domain represents a finite region of space which is infinite space.

This condition should allow the wave **leave** the computational domain.

# Example without radiation boundary condition




(c) time=23

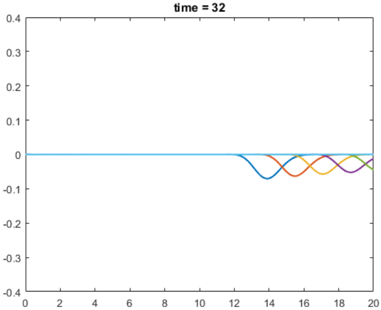



(d) time=24

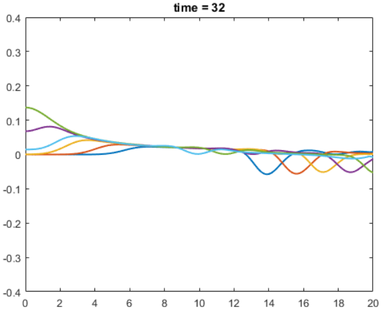
Note the axis scale.

# Example without radiation boundary condition

Gaussian pulse in spherically symmetric coordinates.




(e) with boundary condition




(f) without boundary condition

# Maximally dissipative boundary condition

Assume that a boundary condition form

$$w_-|_{\partial\Omega} = S\ w_+|_{\partial\Omega}$$

Choose a matrix $S$ small enough to lead the numerical scheme well-posed. That is, the energy bounded,

$$E(t) = \int_\Omega \langle u, Hu\rangle\, dV\,.$$

**Radiation boundary condition or Sommerfeld condition**

Choose $S = 0$ on boundary.

No incoming wave. The field satisfy $\Pi + \Phi = 0$ outgoing wave.

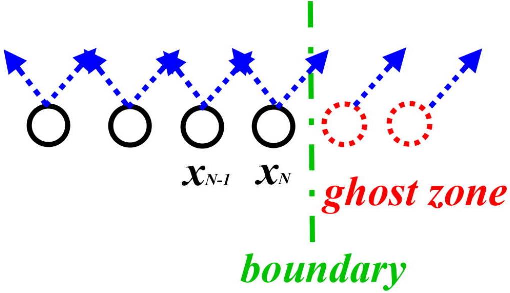
# Constraints?

**Problem?**

Is the evolution solution at boundary satisfy constraints?

Don't worry! We did not solve the other variables $\{a, \alpha\}$ by evolution but by constrain equations.

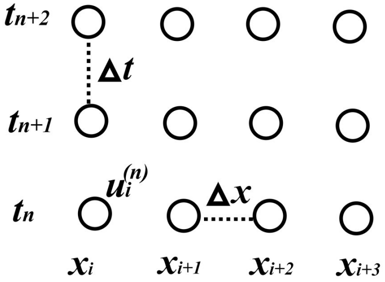
# Implement radiation boundary condition



# Outline

# Notations

Before introducing the numerical scheme, some notations have to be claims,




Note that $u^{(n)}$ denotes the time step $n$, while $u^{[i]}$ denotes sub-step $i$. $\mathcal{S}$ is spatial difference approximation.

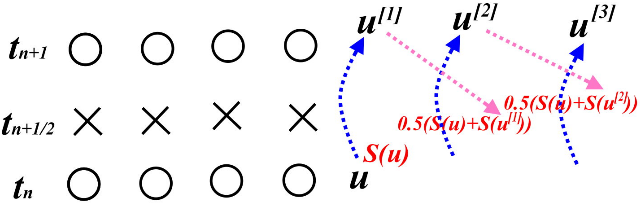
# Iterative Crank-Nicholson (ICN) method

Consider PDE $u_t = u_x$.

$$u^{*[1]} = u^{(n)} + \Delta t S(u^{(n)})$$

$$u^{*[k]} = u^{(n)} + \frac{\Delta t}{2}\left[S(u^{(n)}) + S(u^{*[k-1]})\right], \quad \text{for } k = 2, \cdots, K$$

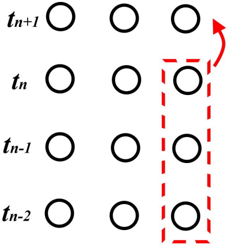
$$u^{(n+1)} = u^{*[K]}.$$



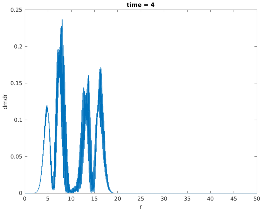

- Pros: adaptive time steps, save memory.
- Cons: "sneaking" $a$, $\alpha$ on next time step take much time.

# Original method

$$y^{(n+3)} = y^{(n+2)} + h\left(\frac{23}{12}\mathcal{S}(y^{(n+2)}) - \frac{16}{12}\mathcal{S}(y^{(n+1)}) + \frac{5}{12}\mathcal{S}(y^{(n)})\right)$$



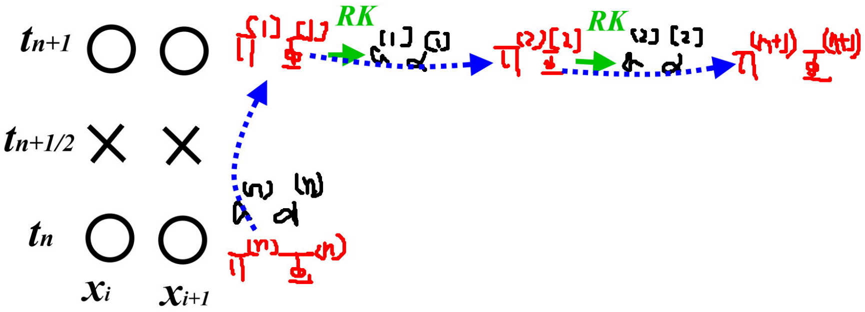

(g) multistep method




(h) result of multistep method

Note that two method is different between explicit and implicit methods.

# Implement of ICN method



# Why ICN method?

- Adaptive time
- Replace derivative w.r.t. time for radius. Consider the PDE $u_t = a(x)u_x$, then

$$u(x,t_{n+1}) = u(x,t_n) + \Delta t u_t(x,t_n) + \frac{\Delta t^2}{2} u_{tt}(x,t_n) + O(\Delta t^3)$$
$$= u(x,t_n) + \Delta t a u_x(x,t_n) + \frac{\Delta^2}{2} a\left(a u_x\left(x,t_n\right)\right)_x + O(\Delta t^3)$$

However, $a$ is dependent on time and space. It is difficult to implement this method.

# Motivation of artificial dissipative term

The following is known as Lax-Wendroff schem. It is conditionally stable.

$$u_j^{(n+1)} = u_j^{(n)} + \frac{c\Delta t}{2\Delta x}(u_{j+1}^{(n)} - u_{j-1}^{(n)}) + \frac{1}{2}\left(u_{j+1}^{(n)} - 2u_j^{(n)} + u_{j-1}^{(n)}\right).$$

The last term is diffusion term. Hence, we want to add artificial term to damp the high frequency oscillation.

# generalize Lax Wendroff (LW) scheme

LW scheme can be modified by adding the term of form

$$u_j^{(n+1)} = u_j^{(n)} + \Delta t \mathcal{S}(u_j^{(n)}) - \epsilon \frac{\Delta t}{\Delta x}(-1)^N \Delta_x^{2N} u_j^{(n)},$$

where $\epsilon > 0$, $N \in \mathbb{N}$ and $\Delta_x^{2N}$ that mimic to the high order derivatives $\partial_x^{2N} u$ is defined by

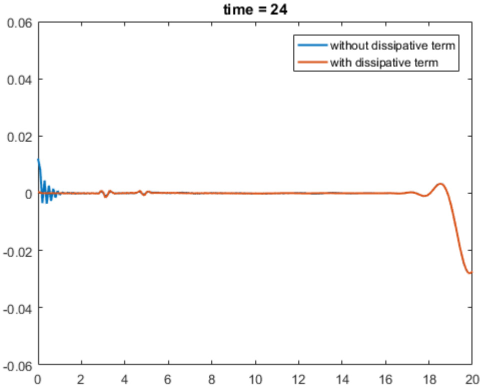
$$\Delta_x^2 u_j^{(n)} = u_{j+1}^{(n)} - 2u_j^{(n)} + u_{j-1}^{(n)}$$
$$\Delta_x^4 u_j^{(n)} = u_{j+2}^{(n)} - 4u_{j+1}^{(n)} + 6u_j^{(n)} - 4u_{j-1}^{(n)} + u_{j-2}^{(n)}$$

## Remark

This term is $O((\Delta x)^{2N-1})$. Hence, if the second order difference is chosen, the artificial dissipative term is $\Delta_x^4$, *i.e.* $N = 2$.

# Comparison between with and without dissipative

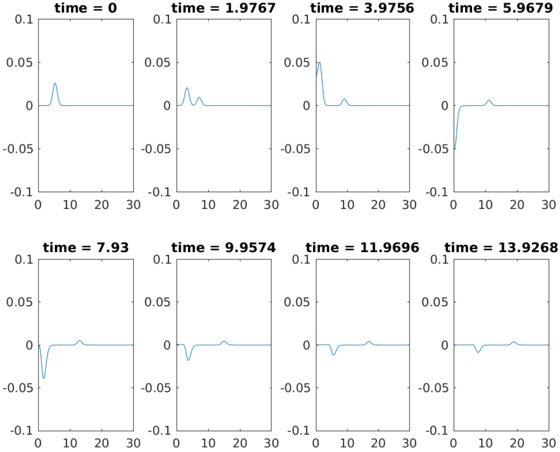
Orange line represents without dissipative term; blue line represents with dissipative term.



# Outline

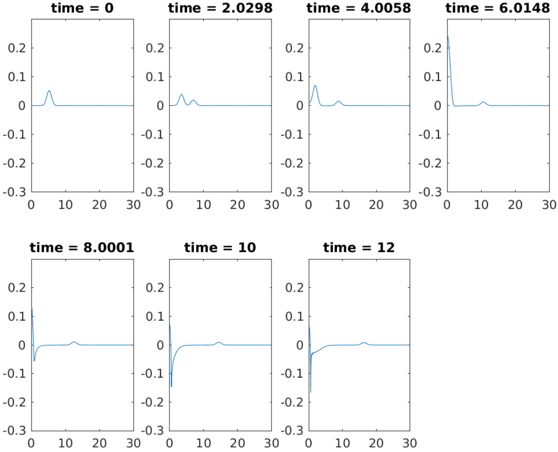
# Weak scalar field

Initial scalar field is $\psi_A(r) = Ar^2 e^{-(r-5)^2}$, where $A = 10^{-3}$

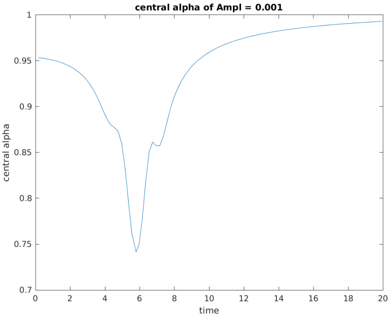


# Strong scalar field

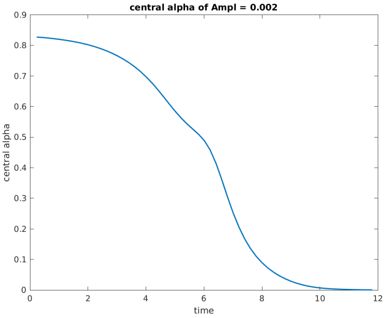
Initial scalar field is $\psi_A(r) = Ar^2 e^{-(r-5)^2}$, where $A = 2 \times 10^{-3}$



# Lapse function collapse



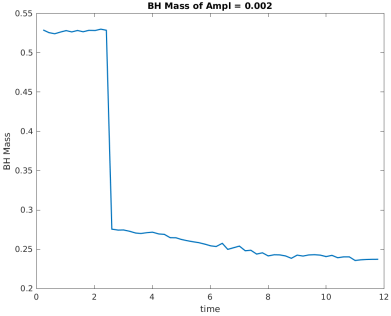

(i) Central value of $\alpha$ for weak field



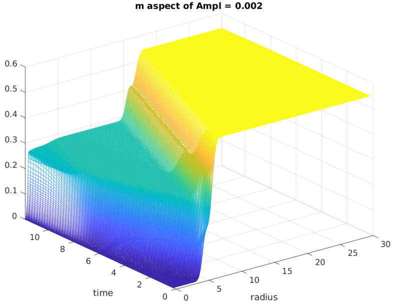

(j) Central value of $\alpha$ for strong field

# Observation of mass

The ADM mass $M_{ADM}$ in computational domain is about 0.54. Then, the mass aspect of black hole is about 0.24 in stable region.




(k) Mass aspect at maximum of $\frac{2m}{r}$




(l) Mass aspect

# Outline

# Summary

- Characteristic analysis
- Radiation Boundary Conditions
- Iterative Crank-Nicholson method
- Confirm critical phenomena

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[1] M. Alcubierre, B. Bruegmann, T. Dramlitsch, J. A. Font, P. Papadopoulos, E. Seidel, N. Stergioulas and R. Takahashi, *Towards a Stable Numerical Evolution of Strongly Gravitating Systems in General Relativity: The Conformal Treatments* Phys. Rev. D, (2000).

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